MATRIX COEFFICIENTS OF UNITARY REPRESENTATIONS AND ASSOCIATED COMPACTIFICATIONS

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Abstract. We study, for a locally compact group G, the compactifications (π, G^π) associated with unitary representations π, which we call π-*Eberlein compactifications*. We also study the Gelfand spectra $\Phi_{\mathcal{A}(\pi)}$ of the uniformly closed algebras $\mathcal{A}(\pi)$ generated by matrix coefficients of such π. We note that $\Phi_{\mathcal{A}(\pi)} \cup \{0\}$ is itself a semigroup and show that the Šilov boundary of $\mathcal{A}(\pi)$ is G^π. We study containment relations of various uniformly closed algebras generated by matrix coefficients, and give a new characterisation of amenability: the constant function 1 can be uniformly approximated by matrix coefficients of representations weakly contained in the left regular representation if and only if G is amenable. We show that for the universal representation ω, the compactification (ω, G^ω) has a certain universality property: it is universal amongst all compactifications of G which may be embedded as contractions on a Hilbert space, a fact which was also recently proved by Megrelishvili [48]. We illustrate our results with examples including various abelian and compact groups, and the $ax+b$-group. In particular, we witness algebras $\mathcal{A}(\pi)$, for certain non-self-conjugate π, as being generalised algebras of analytic functions.

1. Preliminaries

1.1. **Introduction.** Given a locally compact group G, introverted subspaces of $\mathcal{CB}(G)$, the continuous bounded functions on G, and their associated compactifications of G have been studied by many authors over the years; see, for example, the treatise of Berglund, Junghenn and Milnes [6]. Let $\mathrm{B}(G)$ denote the Fourier-Stieltjes algebra of G and $\mathcal{E}(G)$ its uniform closure in $\mathcal{CB}(G)$, which we call the *Eberlein algebra*. The algebras of left uniformly continuous functions $\mathcal{LUC}(G)$, weakly almost periodic functions $\mathcal{WAP}(G)$ and almost periodic functions $\mathcal{AP}(G)$, and their associated compactifications $G^{\mathcal{LUC}}$, $G^{\mathcal{WAP}}$ and $G^{\mathcal{AP}}$, have received a great deal of attention over the years, while less has been paid to $\mathcal{E}(G)$ and $G^{\mathcal{E}}$. Being a quotient of $\mathcal{WAP}(G)^*$, $\mathcal{E}(G)^*$ inherits many of the nicest properties of $\mathcal{WAP}(G)^*$, in particular Arens regularity. In some situations $\mathcal{E}(G) = \mathcal{WAP}(G)$; for connected G this has been characterised by Mayer [45].

In the present article we initiate a comprehensive investigation into the properties of $\mathcal{E}(G)$, $\mathcal{E}(G)^*$ and the associated compactification $G^{\mathcal{E}}$. This is a natural task as Eberelein [15] initiated the theory of weakly almost periodic functions on

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abelian groups in order to gain understanding of the Fourier-Stieltjes transforms of measures on the dual group. We note that the spectrum $\Phi_{\mathrm{B}(G)}$ of $\mathrm{B}(G)$ can be very complicated, especially for abelian groups — see the monographs of Rudin [56] and Graham and McGehee [26]. Meanwhile, the spectrum of $\mathcal{E}(G)$, being the closure of G in $\Phi_{\mathrm{B}(G)}$, is much more tractable. In order to be systematic, we restrict ourselves not only to examining $\mathcal{E}(G)$, but, in fact uniform algebras $\mathcal{A}(\pi)$ generated by matrix coefficients of general unitary representations π. Herein we gain some extra complications and generalise, in a certain manner, the theory of "generalised algebras of analytic functions" in the sense of Arens and Singer [1]. We find, for example, that the Šilov boundary of $\mathcal{A}(\pi)$ is $G^{\mathcal{A}(\pi)}\setminus\{0\}$, where $G^{\mathcal{A}(\pi)}$ is the associated compactification of G.

One of the most surprising features is that we can use semigroup structure theory to gain an analogue of the Fell-Hulanicki characterisation of amenability. We also show that $G^{\mathcal{E}}$ is the universal compactification amongst those compactifications of G which are realisable as contractions on a Hilbert space, a result already proved by Megrelishvili [48], although with different techniques. An interesting feature which arises is that there are involutive compactifications of G, i.e. admiting an involution $x\mapsto x^*$ which extends $s\mapsto s^{-1}$, which cannot faithfully represented as contractions on a Hilbert space.

The second named author owes his interest in Eberlein compactifications to his work on group algebra homomorphism problems. In [59], it is established that $*$-homomorphisms of $L^1(G)$ into $M(H)$ are in bijective correspondence with weak*-continuous $*$-homomorphisms from $M(G^{\mathcal{E}})$ into $M(H^{\mathcal{E}})$, and from $M(G^{\mathcal{E}})$ into $M(H)$.

1.2. **Plan.** While our focus and goal is to understand these certain function spaces for locally compact groups we have realised that much of the general theory can be framed in the much more general context of a semitopological semigroup. Hence in Section 2 we study spaces of functions over a semi-topological semigroup, which we need not even assume is locally compact. The philosophy of this section is that of [6], in which the duality between certain translation-invariant unital C*-subalgebras of functions and compactifications is the major tool. We augment this in a modest but critical manner. Since our goal is to understand compactifications associated to matrix coefficients, as introduced in §§1.3, we find it handy to use the notion of homogeneous spaces of functions on a semigroup, which we describe in §§2.2. We give a systematic exposition of the basic theory of such subspaces and discuss the algebras and self-adjoint algebras generated by them. In §§2.3 we focus on semitopological compactifications. Particularly, we observe that if the underlying semigroup G itself has a continuous involution — say $s\mapsto s^{-1}$ in the case of a topological group — then the weakly almost periodic compactification is universal amongst compactifications which themselves admit a continuous involution extending that of G. In §§2.4 we introduce the concept of (CH)-compactifications, those realisable as weak*-closed semigroups of contractions on a Hilbert space.

The heart of this article is Section 3. We return to the setting of a locally compact group G. We study the relationships between various spaces generated by matrix coefficients, both uniformly closed and closed in the Fourier-Stieltjes norm. We introduce the concept of *Eberlein containment* and illustrate its relationship to more classical methods of comparing unitary representations. In particular we observe that the Eberlein compactification, which is the spectrum of the uniform

closure of the Fourier-Stieltjes algebra in $\mathcal{CB}(G)$, is an invariant for G. In §§3.2 we study the spectra of algebras generated by matrix coefficients, giving a criterion for determining elements of the spectra of these algebras in the vein of Walter [64, 63], and characterising the Šilov boundary within the spectrum. In §§3.3 we manipulate the role played by almost periodic functions to characterise amenability of G in terms of some uniformly closed algebras of functions. In §§3.4 we recover a result already shown by Megrelishvili [48], that describes a natural universality property of the Eberlein compactification. As it is further noted in [48], there exist monothetic compact semitopological semigroups which are not (CH)-compactifications. We extend this observation to include wider classes of groups, using results of Chou [9] and Mayer [45, 46].

In Section 4 we illustrate aspects of our theory with examples. We include spine-type examples, after Ilie and the first named author [36] and Berglund [5], however we modify them to show special properties of the compactifications. We show how abelian groups fit into our theory and even compute spectra for subsemigroups of integers and open subsemigroups in vector groups. This emphasises that the non-self-adjoint algebras of matrix coefficients are indeed "generalised algebras of analytic functions" in the sense of [1]. We continue on this track by illustrating an example for compact groups, and finally the $ax+b$-group.

1.3. **The basic spaces.** We consider several spaces of functions based upon unitary representations of a locally compact group G. We let Σ_G denote the class of all continuous unitary representations π, continuous in the sense that each matrix coefficient function, $s \mapsto \langle \pi(s)\xi|\eta\rangle$, is continuous on G. For two elements π, σ of Σ_G, we write $\pi \cong \sigma$ to denote the relation of unitary equivalence. We let $\{\bar{\pi}, \bar{\mathcal{H}}_\pi\}$ denote the conjugate representation. As in [2], for π in Σ_G we define

$$F_\pi = \operatorname{span}\{\langle \pi(\cdot)\xi|\eta\rangle : \xi, \eta \in \mathcal{H}_\pi\}.$$

We observe that F_π is clearly left and right translation invariant and that

$$F_\pi^\vee = \bar{F}_\pi = F_{\bar{\pi}} \tag{1.1}$$

where $\check{u}(s) = u(s^{-1})$. Indeed $\langle \pi(\cdot)\xi|\eta\rangle^\vee = \overline{\langle \pi(\cdot)\eta|\xi\rangle} = \left\langle \bar{\pi}(\cdot)\bar{\eta}|\bar{\xi}\right\rangle$ for ξ, η in $\mathcal{H}_\pi$. Hence F_π is inversion-invariant if $\pi \cong \bar{\pi}$.

We let $\mathrm{B}(G) = \bigcup_{\pi\in\Sigma_G} F_\pi$ denote the *Fourier-Stieltjes algebra* of G as defined in [17]. This is an algebra of functions and, moreover a Banach algebra when endowed with the norm admitting the two equivalent descriptions below

$$\begin{aligned}\|u\|_{\mathrm{B}} &= \inf\left\{\|\xi\|\,\|\eta\| : u = \langle \pi(\cdot)\xi|\eta\rangle, \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\right\} \\ &= \sup\left\{\left|\int_G u(s)f(s)ds\right| : f \in \mathrm{L}^1(G), \|f\|_* = \sup_{\pi\in\Sigma_G} \|\pi_1(f)\| \leq 1\right\}\end{aligned}$$

where $\pi_1(f) = \int_G f(s)\pi(s)ds$ (weak* integral). We note that we obtain the duality identification $\mathrm{B}(G) \cong \mathrm{C}^*(G)^*$, where $\mathrm{C}^*(G) = \overline{\mathrm{L}^1(G)}^{\|\cdot\|_*}$, the completion of $\mathrm{L}^1(G)$ in the largest C*-norm $\|\cdot\|_*$. Let $\mathrm{P}_1(G)$ denote the set of continuous, positive definite functions u with $u(e) = 1$. Then the Gelfand-Naimark construction provides for each u in $\mathrm{P}_1(G)$ a unitary representation π_u and a norm one cyclic vector ξ in $\mathcal{H}_{\pi_u}$ for which $u = \langle \pi_u(\cdot)\xi|\xi\rangle$. Moreover, every such cyclic representation $\{\pi, \xi\}$ arises, up to unitary equivalence, in this manner. See [21, (3,20)] or [14, 13.4.5] for

details. We define the *universal representation*

$$\omega_G = \bigoplus_{u \in \mathrm{P}_1(G)} \pi_u.$$

As shown in [17], $\mathrm{B}(G) = \mathrm{span}\mathrm{P}_1(G) = F_{\omega_G}$. Also, it is well-known that $\|(\omega_G)_1(f)\| = \|f\|_*$ for f in $\mathrm{L}^1(G)$.

We fix an element π of Σ_G. We let

$$\mathrm{A}_\pi = \overline{F_\pi}^{\|\cdot\|_{\mathrm{B}}} \quad \text{and} \quad \mathcal{E}_\pi = \overline{F_\pi}^{\|\cdot\|_\infty}$$

which are each closed translation-invariant subspaces of $\mathrm{B}(G)$ and $\mathcal{CB}(G)$, respectively. We then let $\mathrm{alg}(F_\pi)$ denote the algebra of functions generated by F_π and define

$$\mathrm{A}(\pi) = \overline{\mathrm{alg}(F_\pi)}^{\|\cdot\|_{\mathrm{B}}} \quad \text{and} \quad \mathcal{A}(\pi) = \overline{\mathrm{alg}(F_\pi)}^{\|\cdot\|_\infty}.$$

which are translation-invariant closed subalgebras of $\mathrm{B}(G)$ and $\mathcal{CB}(G)$, respectively. Finally, we let

$$\mathrm{E}(\pi) = \overline{\mathrm{alg}(F_\pi + \bar{F}_\pi)}^{\|\cdot\|_{\mathrm{B}}}, \quad \mathcal{E}(\pi) = \overline{\mathrm{alg}(F_\pi + \bar{F}_\pi)}^{\|\cdot\|_\infty}$$
$$\text{and} \quad \mathcal{E}_1(\pi) = \overline{\mathrm{alg}(\mathbb{C}1 + F_\pi + \bar{F}_\pi)}^{\|\cdot\|_\infty}$$

which are translation-invariant, conjugation-invariant closed subalgebras of $\mathrm{B}(G)$ and $\mathcal{CB}(G)$, respectively. We define representations

$$\tau_\pi = \bigoplus_{n \in \mathbb{N}} \pi^{n\otimes} \quad \text{and} \quad \rho_\pi = \bigoplus_{\substack{m,n \in \{0\} \cup \mathbb{N} \\ m+n \geq 1}} \pi^{m\otimes} \otimes \bar{\pi}^{n\otimes}$$

where τ_π is defined on the Hilbertian direct sum $\mathcal{H}_{\tau_\pi} = \ell^2\text{-}\bigoplus_{n\in\mathbb{N}} \mathcal{H}_\pi^{n\otimes_2}$, and $\mathcal{H}_{\rho_\pi}$ is defined similarly. Then we have that

$$\mathrm{A}(\pi) = \mathrm{A}_{\tau_\pi} \quad \text{and} \quad \mathrm{E}(\pi) = \mathrm{A}_{\rho_\pi}.$$

For details see [58, Lem. 4.1]. The fact that $\|\cdot\|_{\mathrm{B}} \geq \|\cdot\|_\infty$ then implies that

$$\mathcal{A}(\pi) = \mathcal{E}_{\tau_\pi}, \quad \mathcal{E}(\pi) = \mathcal{E}_{\rho_\pi} \quad \text{and} \quad \mathcal{E}_1(\pi) = \mathcal{E}_{1\oplus\rho_\pi}.$$

We let B_π denote the weak* closure of F_π in $\mathrm{B}(G)$. We observe that it follows from [17, (1.20)] that the representation $\omega_\pi = \bigoplus_{u\in\mathrm{P}_1(G)\cap\mathrm{B}_\pi} \pi_u$ satisfies $\mathrm{B}_\pi = F_{\omega_\pi}$. We define the *weak π-Eberlein algebra* by

$$\mathcal{EB}(\pi) = \overline{\mathrm{alg}(\mathrm{B}_\pi + \bar{\mathrm{B}}_\pi)}^{\|\cdot\|_\infty} = \mathcal{E}(\omega_\pi).$$

The definition of the weak π-Eberlein algebra is arguably the least natural one here as it mixes topologies; it is motivated by its use in Theorem 3.12. Finally, we let the *Eberlein algebra* of G be given by

$$\mathcal{E}(G) = \mathcal{E}(\omega_G) = \overline{\mathrm{B}(G)}^{\|\cdot\|_\infty}$$

where ω_G is the universal representation, defined above.

2. Function spaces over semigroups and compactifications

For this section we will always let G be a semitopological semigroup, not necessarily locally compact.

2.1. **Arens products and semigroup compactifications.** Our standard reference for this section is the text [6], though our notation differs slightly.

A *right topological compactification* of G is a pair (δ, S), where S is a compact right topological semigroup — for any t in S, $s \mapsto st$ is continuous — and $\delta : G \to S$ is a continuous homomorphism whose range is both dense in S and contained in the topological centre $Z_T(S) = \{t \in G : s \mapsto ts \text{ is continuous}\}$. We define left topological compactifications similarly. If S is semitopological, i.e. $S = Z_T(S)$, then we say (δ, S) is a *semitopological compactification* of G.

If $(\delta, S), (\varepsilon, T)$ are two right [left] topological compactifications of G we write $(\delta, S) \leq (\varepsilon, T)$ if there is a continuous homomorphism $\theta : T \to S$ such that $\theta \circ \varepsilon = \delta$; necessarily, θ is surjective. We say (δ, S) is a *factor* of (ε, T), conversely (ε, T) is an *extension* of (δ, S). We say (δ, S) and (ε, T) are equivalent, written $(\delta, S) \cong (\varepsilon, T)$ if θ, above, is an isomorphism. This condition is the same as simultaneously having $(\delta, S) \leq (\varepsilon, T)$ and $(\delta, S) \geq (\varepsilon, T)$. In particular, $\leq$ is a partial ordering on the class, in fact the set (see the remark after Theorem 2.1, below) of equivalence classes of right [left] topological compactifications of G.

Let $\mathcal{CB}(G)$ denote the C*-algebra of continuous complex-valued bounded functions on G with uniform norm $\|\cdot\|_\infty$. If $f \in \mathcal{CB}(G)$, $s \in G$ we denote the anti-action of left translation and the action of right translation of s on f by

$$f \cdot s(t) = f(st) \text{ and } s \cdot f(t) = f(ts)$$

for t in G. Let $\mathcal{X}$ be a closed linear subspace of $\mathcal{CB}(G)$. We say $\mathcal{X}$ is *left [right] introverted* if it is closed under left [right] translations, and for any m in $\mathcal{X}^*$, $m \cdot f$, defined by

$$m \cdot f(s) = m(f \cdot s) \quad [f \cdot m(s) = m(s \cdot f)]$$

is also an element of $\mathcal{X}$. Note that we do not insist that $\mathcal{X}$ contains the constant functions. We say that $\mathcal{X}$ is *introverted* if it is both left and right introverted.

If $\mathcal{X}$ is left [right] introverted then we define the *left [right] Arens products* on $\mathcal{X}^*$ by

$$m \Box n(f) = m(n \cdot f) \quad [m \Diamond n(f) = n(f \cdot m)] \quad \text{for } f \text{ in } \mathcal{X} \tag{2.1}$$

which makes $\mathcal{X}^*$ into a right [left] dual Banach algebra in the sense that for a fixed n, $m \mapsto m \Box n$ $[m \mapsto n \Diamond m]$ is weak*-weak* continuous on $\mathcal{X}^*$. If $\mathcal{X}$ is introverted, we say that $\mathcal{X}$ is *Arens regular* if the left and right Arens products coincide. Arens regularity is discussed in greater detail in Section 2.3.

If $\mathcal{X}$ is a closed subalgebra of $\mathcal{CB}(G)$, then we denote its Gelfand spectrum by $\Phi_{\mathcal{X}}$, and endow it with its weak* topology. We let $\varepsilon_{\mathcal{X}} : G \to \Phi_{\mathcal{X}}$ denote the evaluation map, which has dense range in the case that $\mathcal{X}$ is a C*-algebra. We say that $\mathcal{X}$ is *left [right] m-introverted* if $\chi \cdot f \in \mathcal{X}$ $[f \cdot \chi \in \mathcal{X}]$ for each χ in $\Phi_{\mathcal{X}}$. We record, for ease of reference, the following standard result which can be found as Theorems 3.1.7 and 3.1.9 in [6].

Theorem 2.1. (i) *Let $\mathcal{X}$ be a left [right] translation invariant unital C*-subalgebra of $\mathcal{CB}(G)$. Then $\mathcal{X}$ is left [right] m-introverted if and only if $\Phi_{\mathcal{X}}$ is a right [left] topological semigroup under the left [right] Arens product of (2.1). In this case $(\varepsilon_{\mathcal{X}}, \Phi_{\mathcal{X}})$ is a right [left] topological compactifiaction of G and $\mathcal{X} = \mathcal{C}(\Phi_{\mathcal{X}}) \circ \varepsilon_{\mathcal{X}}$.*

(ii) *If $(\delta, S), (\varepsilon, T)$ are two right [left] topological compactifications of G then*

$$(\delta, S) \leq (\varepsilon, T) \quad \textit{if and only if} \quad \mathcal{C}(S) \circ \delta \subset \mathcal{C}(T) \circ \varepsilon.$$

In particular, the family of all equivalence classes of right [left] topological compactifications of G is a set, realised in bijective correspondence with the left [right] m-introverted unital C*-subalgebras of $\mathcal{CB}(G)$.

2.2. **Homogeneous subspaces and the algebras they generate.** Let us consider a mild generalisation of the concept of introverted subspaces of $\mathcal{CB}(G)$, which will be useful for our goals. For the sake of brevity we will work mainly with left actions on subspaces and hence right topological dual algebras and compactifications; the opposite handed analogues are similar. A *left homogeneous subspace* of $\mathcal{CB}(G)$ is a subspace X such that

(i) X is equipped with a norm $\|\cdot\|$ under which it is complete and for which $\|f\| \geq \|f\|_\infty$ for f in X; and
(ii) X is left translation invariant and $(f, s) \mapsto f\cdot s : X\times G \to X$ is continuous in s and contractive in f.

Moreover we say that X is *left introverted* if, further

(iii) $M\cdot f \in X$ for M in X^*, with $\|M\cdot f\| \leq \|M\| \, \|f\|$.

Notice that as an immediate consequence of (i), the evaluation functionals $\varepsilon_X(s)$, for s in G, are bounded; moreover the family $\varepsilon_X(G) = \{\varepsilon_X(s)\}_{s\in G}$ is separating. We observe that if we assume, in place of (iii), only that $M\cdot X \subset X$ for a fixed M in X^* then it is automatic that the introversion operator $f \mapsto M\cdot f$ is bounded. Indeed, we appeal to the closed graph theorem: if $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} M\cdot f_n = g$, then for any s in G we have $g(s) = \lim_{n\to\infty} M\cdot f_n(s) = \lim_{n\to\infty} M(f_n\cdot s) = M(f\cdot s) = M\cdot f(s)$ and hence $g = M\cdot f$. Similarly, if $X^*\cdot f \subset X$, for a fixed f in X, then $M \mapsto M\cdot f$ is bounded. However, we are aware of no means by which to prove that the map $(M, f) \mapsto M\cdot f$ is contractive.

If X and Y are both left homogeneous Banach spaces in $\mathcal{CB}(G)$, we say $X \subset Y$ boundedly (contractively) if X is a subspace of Y, and the inclusion map $X \hookrightarrow Y$ is bounded (contractive).

Proposition 2.2. *Let X be a left introverted homogeneous subspace of $\mathcal{CB}(G)$. Then:*

(i) *X^* is a right dual Banach algebra under the left Arens product;*
(ii) *$\mathcal{X} = \overline{X}^{\|\cdot\|_\infty}$ is left introverted with $X \subset \mathcal{X}$ contractively; and*
(iii) *if Y is another left introverted homogeneous Banach space in $\mathcal{CB}(G)$, then $X \subset Y$ boundedly (contractively) if and only if there is weak*-weak* continuous (contractive) operator $\Phi : Y^* \to X^*$ such that $\Phi(\varepsilon_Y(s)) = \varepsilon_X(s)$ for s in G. The operator Φ is necessarily a homomorphism (with respect to left Arens product). If X is a closed subspace of Y then Φ is a quotient map.*

Proof. (i) This is standard, but short, so we include a full proof for completeness. We have for M, N in X^* that $\|(M\square N)(f)\| \leq \|M\| \, \|N\cdot f\| \leq \|M\| \, \|N\| \, \|f\|$ for f in X, so $\|M\square N\| \leq \|M\| \, \|N\|$. Associativity remains: if $L, M, N \in X^*$ and $f \in X$ then

$$(M\square N)\cdot f(s) = M\square N(f\cdot s) = M(N\cdot(f\cdot s)) = M((N\cdot f)\cdot s) = M\cdot(N\cdot f)(s)$$

for s in G and hence

$$L\square(M\square N)(f) = L((M\square N)\cdot f) = L(M\cdot(N\cdot f)) = (L\square M)(N\cdot f) = (L\square M)\square N(f).$$

It is clear that $M \mapsto M\square N$ is weak* continuous, so X^* is a left dual Banach algebra.

(ii) If $m \in \mathcal{X}^*$ and $f \in X$, then $m\cdot f = M\cdot f \in X$ where $M = m|_X$. We note that $\|m\cdot f\|_\infty \leq \|m\| \sup_{s\in G} \|f\cdot s\|_\infty \leq \|m\| \, \|f\|_\infty$. By density of X in $\mathcal{X}$ and continuity of $f \mapsto m\cdot f$ it follows that $\mathcal{X}$ is left introverted.

(iii) If $X \subset Y$ boundedly, we let $\Phi : Y^* \to X^*$ be the adjoint of the inclusion map, which is the restriction map. Then Φ intertwines ε_Y and ε_X. For f in X and N in Y^* we have

$$N\cdot f(s) = N(f\cdot s) = \Phi(N)(f\cdot s) = \Phi(N)\cdot f(s)$$

for s in G so $N\cdot f \in X$. Then if, further, $M \in X^*$, we obtain

$$\Phi(M\Box N)(f) = M\Box N(f) = M(N\cdot f) = \Phi(M)(\Phi(N)\cdot f) = \Phi(M)\Box\Phi(N)(f).$$

Conversely, if there exists a weak*-weak* continuous operator $\Phi : Y^* \to X^*$ which intertwines ε_X and ε_Y, then the pre-adjoint $\varphi : X \to Y$ of Φ must satisfy $\varphi(f)(s) = \varepsilon_Y(s)(\varphi(f)) = \Phi(\varepsilon_Y(s))(f) = \varepsilon_X(s)(f) = f(s)$. Thus $X \subset Y$ boundedly and φ is the inclusion map. If X is a closed subspace of Y, then $\Phi : Y^* \to X^*$, being the restriction map, is a quotient map by the Hahn-Banach theorem. $\Box$

We consider a mild generalisation of Theorem 2.1 (i).

Proposition 2.3. *Let X be a left introverted homogeneous subspace of $\mathcal{CB}(G)$, $\varepsilon_X : G \to X^*$ be the evaluation map and*

$$G^X = \overline{\varepsilon_X(G)}^{w^*} \subset X^*. \tag{2.2}$$

If G^X is endowed with the weak topology, then (ε_X, G^X) is a right topological compactification of G.*

We call (ε_X, G^X) the *X-compactification* of G.

Proof. This proof is similar to an aspect of that of Theorem 2.1, but short. We first note that

$$\varepsilon_X(G) \subset Z_T(X^*) = \{M \in X^* : N \mapsto M\Box N \text{ is weak*-weak* continuous}\}.$$

Indeed, if $s \in G$ and $N \in X^*$ we have for f in X that $\varepsilon_X(s)\Box N(f) = \varepsilon_X(s)(N\cdot f) = N(f\cdot s)$, so $N \mapsto \varepsilon_X(s)\Box N$ is weak*-weak* continuous. Now if if $\chi, \chi' \in G^X$, we let $\chi = \text{weak*-}\lim_\alpha \varepsilon_X(s_\alpha)$ and $\chi' = \text{weak*-}\lim_\beta \varepsilon_X(t_\beta)$ for nets $(s_\alpha), (t_\beta)$ from G, and we have

$$\chi\Box\chi' = \lim_\alpha \varepsilon_X(s_\alpha)\Box\chi' = \lim_\alpha \lim_\beta \varepsilon_X(s_\alpha)\varepsilon_X(t_\beta) = \lim_\alpha \lim_\beta \varepsilon_X(s_\alpha t_\beta) \in G^X.$$

Being a subsemigroup of the right dual Banach algebra X^* (see Proposition 2.2 (i)), G^X itself is right topological. $\Box$

We now consider two closed subalgebras of $\mathcal{CB}(G)$ generated by a left [right] introverted homogeneous subspace X:

$$\mathcal{A}(X) = \overline{\text{alg}(X)}^{\|\cdot\|_\infty}, \quad \mathcal{E}(X) = \overline{\text{alg}(X + \bar{X})}^{\|\cdot\|_\infty} \quad \text{and} \quad \mathcal{E}_1(X) = \mathcal{E}(X) + \mathbb{C}1$$

where $\bar{X}$ denotes the space of complex conjugates of elements of X and $\text{alg}(X + \bar{X})$ the algebra generated by elements in X and $\bar{X}$. We note that $\mathcal{E}_1(X) = \mathcal{E}(X)$ if the latter is unital, and is the C*-unitization otherwise.

Theorem 2.4. *Let X be a left introverted homogeneous subspace of $\mathcal{CB}(G)$. Then*

(i) *$\mathcal{E}(X)$ and $\mathcal{E}_1(X)$ are left m-introverted.*

(ii) *$(\varepsilon_{\mathcal{E}_1(X)}, \Phi_{\mathcal{E}_1(X)}) \cong (\varepsilon_X, G^X)$ as compactifications of G, and $\Phi_{\mathcal{E}(X)}$ is homeomorphic to $G^X \setminus \{0\}$; and*

(iii) $0 \in G^X \Leftrightarrow 1 \notin \mathcal{E}(X)$.

Proof. If $\chi \in \Phi_{\mathcal{E}(X)}$ let $\chi' = \chi|_X \in X^*$. We have for a polynomial p in $n+m$ variables with $p(0)=0$, $f_1,\dots,f_n,g_1,\dots,g_m$ in X, and s in G, that

$$\begin{aligned}\chi\cdot p(f_1,\dots,\bar g_m)(s) &= \chi(p(f_1\cdot s,\dots,\overline{g_m\cdot s}))\\ &= p(\chi'(f_1\cdot s),\dots,\overline{\chi'(g_m\cdot s)}) = p(\chi'\cdot f_1,\dots,\overline{\chi'\cdot g_m})(s)\end{aligned} \tag{2.3}$$

so $\chi\cdot p(f_1,\dots,\bar g_m) \in \mathrm{alg}(X+\bar X)$. The introversion operator $f \mapsto \chi\cdot f : \mathcal{E}(X) \to \ell^\infty(G)$ is contractive, and takes a dense subspace of $\mathcal{E}(X)$ into $\mathcal{E}(X)$, hence $\mathcal{E}(X)$ is m-introverted. If $\chi \in \Phi_{\mathcal{E}_1(X)}$, then $\chi(1)=1$ and the argument above, applied to an arbitrary polynomial, shows that $\mathcal{E}_1(X)$ is m-introverted. Hence (i) is proved.

We shall prove (ii) and (ii) simultaneously. Since $\mathcal{E}_1(X)$ is a unital C*-algebra of $\mathcal{CB}(G)$, we have that $\Phi_{\mathcal{E}_1(X)} = G^{\mathcal{E}_1(X)}$. The restriction map $\theta : \Phi_{\mathcal{E}_1(X)} \to G^X$, $\theta(\chi) = \chi|_X$, is weak*-weak* continuous which satisfies $\theta\circ\varepsilon_{\mathcal{E}_1(X)}(s) = \varepsilon_{\mathcal{E}_1(X)}(s)|_X = \varepsilon_X(s)$ for s in G, so $(\varepsilon_X, G^X) \leq (\varepsilon_{\mathcal{E}_1(X)}, \Phi_{\mathcal{E}_1(X)})$; in particular θ is surjective. The map θ is injective since each character is determined by its behavior on $\mathrm{alg}(X+\bar X)$ and hence on X. Thus θ is a homeomorphism and thus a compactification isomorphism. Now if $1 \in \mathcal{E}(X)$ then $\Phi_{\mathcal{E}(X)} = G^{\mathcal{E}(X)}$. Just as above, the map $\theta : \Phi_{\mathcal{E}(X)} \to G^X$ is surjective. If it were the case that $0 \in G^X$, then for some χ in $\Phi_{\mathcal{E}(X)}$ $\chi|_X = 0$, which would imply that $\chi(\mathrm{alg}(X+\bar X)) = \{0\}$, and imply that $\chi = 0$, which is absurd. If $1 \notin \mathcal{E}(X)$, then the unique character χ_∞ on $\mathcal{E}_1(X)$ which annihilates $\mathcal{E}(X)$ satisfies $\theta(\chi_\infty) = 0$, and thus, from the surjectivity of θ, $0 \in G^X$. In this case θ establishes a homeomorphism from $\Phi_{\mathcal{E}(X)} \cong \Phi_{\mathcal{E}_1(X)} \setminus \{\chi_\infty\}$ onto $G^X \setminus \{0\}$. □

We observe that it is possible that $G^X \setminus \{0\}$ is not a subsemigroup of G^X. If $G = \{o, e_1, e_2\}$ is the semilattice which is generated by the relation $e_1e_2 = o$, then $\mathcal{X} = \{f \in \mathcal{C}(G) : f(o) = 0\}$ is an introverted subspace, in fact a subalgebra, for which $G^{\mathcal{X}} = \varepsilon_{\mathcal{X}}(G) \cong G$ and $0 = \varepsilon_{\mathcal{X}}(o) \in G^{\mathcal{X}}$. Clearly $G^{\mathcal{X}} \setminus \{0\} \cong G \setminus \{o\}$. A related example, where G is a group, is given in Example 4.2, below.

The following is immediate from Theorem 2.1.

Corollary 2.5. *If X, Y are two left homogeneous introverted subspaces of $\mathcal{CB}(G)$ then $(\varepsilon_X, G^X) \leq (\varepsilon_Y, G^Y)$ if and only if $\mathcal{E}(X) \subset \mathcal{E}_1(Y)$. In particular $(\varepsilon_X, G^X) \cong (\varepsilon_Y, G^Y)$ if and only if $\mathcal{E}_1(X) = \mathcal{E}_1(Y)$*

We obtain an augmentation of Theorem 2.1 (i). An open subset U of a right topological semigroup S has *relatively proper right translations* if for every element s in U and every compact subset K of U, $Ks^{-1} \cap U$ is compact, where $Ks^{-1} = \{t \in S : ts \in K\}$.

Corollary 2.6. *Let $\mathcal{X}$ be a left translation invariant C*-subalgebra of $\mathcal{CB}(G)$. Then $\mathcal{X} = \mathcal{C}_0(G^{\mathcal{X}} \setminus \{0\})\circ\varepsilon_{\mathcal{X}}$. Moreover, $\mathcal{X}$ is left m-introverted if and only if $G^{\mathcal{X}}$ is a right topological semigroup for which $G^{\mathcal{X}} \setminus \{0\}$ has relatively proper right translations.*

Proof. We may and will assume that $1 \notin \mathcal{X}$. Let $\mathcal{X}_1 = \mathbb{C}1 \oplus \mathcal{X}$ be the C*-unitisation of $\mathcal{X}$. From the theorem above we have that $\Phi_{\mathcal{X}_1} = G^{\mathcal{X}_1} \cong G^{\mathcal{X}}$ and $\Phi_{\mathcal{X}} = G^{\mathcal{X}} \setminus \{0\}$. Since $\mathcal{X}_1 = \mathcal{C}(G^{\mathcal{X}})\circ\varepsilon_{\mathcal{X}}$ and 0, in $G^{\mathcal{X}}$, corresponds to the unique character which annihilates $\mathcal{X}$, we have that $\mathcal{X} = \mathcal{C}_0(G^{\mathcal{X}} \setminus \{0\})\circ\varepsilon_{\mathcal{X}}$.

If $\mathcal{X}$ is left m-introverted, then it is straighforward that $\mathcal{X}_1$ is left m-introverted, so $G^{\mathcal{X}} \cong G^{\mathcal{X}_1}$ is a right topological semigroup by Theorem 2.1 (i). The condition

that $G^{\mathcal{X}} \setminus \{0\}$ has relatively proper right translations is equivalent to the condition that

$$\chi \cdot \mathcal{C}_0(G^{\mathcal{X}} \setminus \{0\}) \subset \mathcal{C}_0(G^{\mathcal{X}} \setminus \{0\}) \text{ for every } \chi \text{ in } G^{\mathcal{X}} \setminus \{0\}. \tag{2.4}$$

To see this, we require essentially the proof of [6, Ex. 3.1.10], which we simply adapt to our situation. If $U = G^{\mathcal{X}} \setminus \{0\}$ has relatively proper right translations, then for every $f \in \mathcal{C}_0(U)$, $\varepsilon > 0$ and $\chi \in U$ we have $\{\chi' \in U : \chi \cdot f(\chi') \geq \varepsilon\} = \{\chi' \in G^{\mathcal{X}} : f(\chi') \geq \varepsilon\}\chi^{-1} \cap U$ is compact, hence $\chi \cdot f \in \mathcal{C}_0(U)$. Conversely, if U does not have relatively proper right translations, there is compact $K \subset U$ and $\chi \in U$ for which the closed set $K\chi^{-1} \cap U$ is not compact. Then any compactly supported continuous $f : U \to [0, 1]$ which satisfies $K \subset \{\chi' : f(\chi') = 1\}$ would satisfy that $\chi \cdot f \notin \mathcal{C}_0(U)$, hence (2.4) cannot be satisfied. Clearly, (2.4) is necessary and sufficient for $\mathcal{X} = \mathcal{C}_0(G^{\mathcal{X}} \setminus \{0\}) \circ \varepsilon_{\mathcal{X}}$ to be left m-introverted. □

Let us consider some compactifications of G which decompose with respect to a second semigroup. Suppose H is locally compact right topological semigroup for which there is a homomorphism $\eta : G \to H$ with dense range. We say that H has *proper* right translations if it has relatively proper right translations on itself. A left topological compactification (δ, S) of G is said to be an *(η, H)-compactification* if there is a continuous $\theta : H \to S$ such that $\theta \circ \eta = \delta$; such a θ is necessarily a homomorphism. Moreover, (δ, S) is said to be a *regular (η, H)-compactification* if θ is injective and open. The following result is inspired by [43, Lem. 4.1] and [24, Lem. 1].

Proposition 2.7. *Let H be a locally compact noncompact right topological semigroup with proper right translations, $\eta : G \to H$ be a homomorphism with dense range, and X be a left introverted homogeneous subspace of $\mathcal{CB}(G)$. Then (ε_X, G^X) is a regular (η, H)-compactification if and only if $\mathcal{E}(X) \supset \mathcal{C}_0(H) \circ \eta$ and $\mathcal{C}_0(H) \circ \eta$ is an essential ideal in $\mathcal{E}(X)$.*

In this case there is a semigroup decomposition

$$G^X = (G^X \setminus \theta(H)) \sqcup \theta(H) \tag{2.5}$$

where $\theta : H \to G^X$ is an injective, open homomorphism, $\theta(H)$ is dense in G^X, and $G^X \setminus \theta(H)$ is a closed ideal.

Proof. From Theorem 2.4 (ii) we have that $(\varepsilon_X, G^X) \cong (\varepsilon_{\mathcal{E}(X)}, G^{\mathcal{E}(X)})$. Hence it follows Corollary 2.6, above, that $\mathcal{E}(X) = \mathcal{C}_0(G^X \setminus \{0\}) \circ \varepsilon_X$.

If (ε_X, G^X) is a regular (η, H)-compactification with open injective homomorphism θ, then $\theta(H) \subset G^X \setminus \{0\}$. Indeed, if $\theta(H) \ni 0$, then $\{\theta^{-1}(0)\}$ would be an ideal in H, which would violate that H has proper right translations. Since $\theta(H)$ is open and dense in $G^X \setminus \{0\}$, it follows that $\mathcal{C}_0(\theta(H))$ is an essential ideal in $\mathcal{C}_0(G^X \setminus \{0\})$. We thus have

$$\mathcal{C}_0(H) \circ \eta = \mathcal{C}_0(H) \circ \theta^{-1} \circ \varepsilon_X = \mathcal{C}_0(\theta(H)) \circ \varepsilon_X \subset \mathcal{C}_0(G^X \setminus \{0\}) \circ \varepsilon_X = \mathcal{E}(X)$$

and $\mathcal{C}_0(H) \circ \eta$ is an essential ideal in $\mathcal{E}(X)$. Conversely, if $\mathcal{C}_0(H) \circ \eta$ is an essential ideal in $\mathcal{E}(X) = \mathcal{C}_0(G^X \setminus \{0\}) \circ \varepsilon_X$, then there is an open dense subset $U \subset G^X \setminus \{0\}$ such that $\mathcal{C}_0(H) \circ \eta = \mathcal{C}_0(U) \circ \varepsilon_X$. Thus there is a homeomorphism $\theta : H \to U$. For s in G we have that $\theta(\eta(s)) = \varepsilon_X(s)$, i.e. for each f in $\mathcal{C}_0(G^X \setminus \{0\})$ we have that $f(\theta(\eta(s))) = f(\varepsilon_X(s))$. Hence (ε_X, G^X) is a regular (η, H)-compactification.

In the decomposition (2.5), $\theta(H)$ is a dense open subsemigroup by construction. It remains to show that $G^X \setminus \theta(H)$ is an ideal. We observe that $G^X \setminus \theta(H) = \{\chi \in$

$G^X : \chi|_{\mathcal{C}_0(H)\circ\eta} = 0\}$. Hence if $\chi \in G^X \setminus \theta(H)$ and $f \in \mathcal{C}_0(H)$ then for s in G, $\chi\cdot(f\circ\eta)(s) = \chi(f\circ\eta(s)) = 0$, so $\chi\cdot(f\circ\eta) = 0$. Thus if $\chi' \in G^X$ and $\chi \in G^X \setminus \theta(H)$ we have for $f \in \mathcal{C}_0(H)$, $\chi'\Box\chi(f) = \chi'(\chi\cdot f) = 0$, so $\chi'\Box\chi \in G^X \setminus \theta(H)$. □

The non-self-adjoint situation presents more complications than the self-adjoint one. Since a non-self-adjoint uniform algebra $\mathcal{A}$ on G — i.e. a uniformly closed subalgebra of $\mathcal{CB}(G)$ — may admit spectrum larger than $G^{\mathcal{A}}\setminus\{0\}$, the fact that the closure of the spectrum is a semigroup must be checked. Recall that our definition of left m-introversion applies to all closed subalgebras of $\mathcal{CB}(G)$.

Proposition 2.8. *Suppose $\mathcal{A}$ is a closed left m-introverted subalgebra of $\mathcal{CB}(G)$. Then $\Phi_{\mathcal{A}} \cup \{0\}$ is a semigroup under left Arens product. If $\mathcal{A}$ is unital, then $\Phi_{\mathcal{A}}$ itself is a semigroup.*

As with non-unital self-adjoint algebras, we cannot expect that $\Phi_{\mathcal{A}}$ is a semigroup when $1 \notin \mathcal{A}$.

Proof. If $\chi \in \Phi_{\mathcal{A}}$, $f, g \in \mathcal{A}$ and $s \in G$ then similarly as in (2.3) we have

$$\chi\cdot(fg)(s) = \chi(f\cdot s\, g\cdot s) = \chi(f\cdot s)\chi(g\cdot s) = \chi\cdot f\chi\cdot g(s).$$

Thus if we also have χ' in $\Phi_{\mathcal{A}}$ then

$$\chi\Box\chi'(fg) = \chi(\chi'\cdot(fg)) = \chi(\chi'\cdot f\,\chi'\cdot g) = \chi(\chi'\cdot f)\chi(\chi'\cdot g) = \chi\Box\chi'(f)\chi\Box\chi'(g)$$

so $\chi\Box\chi' \in \Phi_{\mathcal{A}} \cup \{0\}$. If $\mathcal{A}$ is unital, then for $\chi \in \Phi_{\mathcal{A}}$ we have $\chi\cdot 1 = 1$ and it follows for $\chi, \chi' \in \Phi_{\mathcal{A}}$ that $\chi\Box\chi'(1) = 1 \neq 0$. □

If $\mathcal{A}$ is a closed subalgebra of a commutative C*-algebra, then a *boundary* is any closed subset $B \subset \Phi_{\mathcal{A}}$ such that $f \mapsto \hat{f}|_B : \mathcal{A} \to \mathcal{C}_0(B)$ is an isometry ($f \mapsto \hat{f}$ is the Gel'fand transform). The *Šilov boundary* is given by $\partial_{\mathcal{A}} = \bigcap\{B : B$ is a boundary for $\mathcal{A}\}$, and is itself a boundary. See the texts [38, 52] for details including the case that $\mathcal{A}$ is non-unital. In many cases where $\mathcal{A}$ is a uniform algebra on G, $\partial_{\mathcal{A}}$ gives us a means of recovering $G^{\mathcal{A}}$.

Theorem 2.9. *Let X be a left introverted homogeneous subspace of $\mathcal{CB}(G)$.*

(i) *The algebra $\mathcal{A}(X)$ is left m-introverted. The map $\chi \mapsto \chi|_X : \Phi_{\mathcal{A}(X)} \cup \{0\} \to X^*$ is a semigroup homomorphism and homeomorphism onto its range, which takes $G^{\mathcal{A}(X)}$ onto G^X. Moreover $G^{\mathcal{A}(X)} \setminus \{0\}$ is closed in $\Phi_{\mathcal{A}(X)}$.*

(ii) *We have $G^{\mathcal{A}(X)} \setminus \{0\} \supset \partial_{\mathcal{A}(X)}$. Moreover, $\partial_{\mathcal{A}(X)}$ is compact if $1 \in \mathcal{E}(X)$.*

(iii) *If G contains a dense subgroup G_0, then $\overline{\partial_{\mathcal{A}(X)}}^{w*}$ is an ideal in $G^{\mathcal{A}(X)}$. Moreover, if $\varepsilon_X(G_0) \cap \partial_{\mathcal{A}(X)} \neq \varnothing$, then $\partial_{\mathcal{A}(X)} = G^{\mathcal{A}(X)} \setminus \{0\}$.*

The conclusion that $\overline{\partial_{\mathcal{A}(X)}}^{w*}$ is an ideal in $G^{\mathcal{A}(X)} \cong G^X$ is false if we do not assume the existence of G_0. See Example 4.7 (iii), below, for this, and further illustrations. We do not know if the condition $\varepsilon_X(G_0) \cap \partial_{\mathcal{A}(X)} \neq \varnothing$ is automatic if the existence of G_0 is assumed. However, the latter condition is automatic if G itself is of a compact group; see, for example, the proof of [27, 4.2.2].

Proof. That $\mathcal{A}(X)$ is left m-introverted follows a calculation similar to (2.3). Hence it is immediate from Proposition 2.8 that $\Phi_{\mathcal{A}(X)} \cup \{0\}$ is a subsemigroup of $\mathcal{A}(X)^*$. A simple modification of the proof of Theorem 2.4 (ii), above, shows that $\chi \mapsto \chi|_X : \Phi_{\mathcal{A}(X)} \cup \{0\} \to \Phi_{\mathcal{A}(X)}|_X \cup \{0\}$ is a homeomorphism which takes $G^{\mathcal{A}(X)}$ onto G^X. Since $G^{\mathcal{A}(X)}$ is compact, it is closed in $\Phi_{\mathcal{A}(X)} \cup \{0\}$, hence $G^{\mathcal{A}(X)} \setminus \{0\}$ is closed

in $\Phi_{\mathcal{A}(X)}$. The proof of Proposition 2.2 (iii) shows that this restriction map is a semigroup homomorphism. Hence we have (i).

Since $\mathcal{A}(X)$ generates the C*-algebra $\mathcal{E}(X)$, $G^{\mathcal{A}(X)}\setminus\{0\} = (G^{\mathcal{E}(X)}\setminus\{0\})|_{\mathcal{A}(X)} = \Phi_{\mathcal{E}(X)}|_{\mathcal{A}(X)}$ is a boundary for $\mathcal{A}(X)$, so $\partial_{\mathcal{A}(X)} \subset G^{\mathcal{A}(X)}\setminus\{0\}$. Moreover, if $1 \in \mathcal{E}(X) = \mathcal{E}(\mathcal{A}(X))$, then by Theorem 2.4 (iii) we have that $0 \notin G^{\mathcal{A}(X)}$. Hence $G^{\mathcal{A}(X)} = G^{\mathcal{A}(X)}\setminus\{0\}$ is compact, hence so too must be the closed subset $\partial_{\mathcal{A}(X)}$. Hence we have (ii).

Now we consider (iii). If $t \in G_0$ then $f \mapsto \varepsilon_{\mathcal{A}(X)}(t)\cdot f$ and $f \mapsto \varepsilon_{\mathcal{A}(X)}(t^{-1})\cdot f$ are mutually inverse contractions, and hence isometries. We let for t in G_0 and χ in $\Phi_{\mathcal{A}(X)}$, $t\cdot\chi = \varepsilon_{\mathcal{A}(X)}(t)\Box\chi$. Since $\varepsilon_{\mathcal{A}(X)}(G_0) \subset Z_T(G^{\mathcal{A}(X)})$, $\Phi_{\mathcal{A}(X)}$ is a topological G_0-space. If $B \subset \Phi_{\mathcal{A}(X)}$ is any closed boundary, then $t\cdot B$ must also be a closed boundary for any $t \in G_0$. Hence, by minimality, $t\cdot\partial_{\mathcal{A}(X)} \supseteq \partial_{\mathcal{A}(X)}$, and it follows that $t\cdot\partial_{\mathcal{A}(X)} = \partial_{\mathcal{A}(X)}$. Since $\varepsilon_{\mathcal{A}(X)}(G_0)$ is dense in $G^{\mathcal{A}(X)}$ it follows that for χ in $G^{\mathcal{A}(X)}$, $\chi\Box\partial_{\mathcal{A}(X)} \subset \overline{\partial_{\mathcal{A}(X)}}^{w^*}$. Furthermore, if $\varepsilon_X(G_0)\cap\partial_{\mathcal{A}(X)} \neq \varnothing$, then $\varepsilon_X(G_0) \subset \partial_{\mathcal{A}(X)}$, and it follows that $\partial_{\mathcal{A}(X)}$ is dense in $G^{\mathcal{A}(X)}$. Hence $\partial_{\mathcal{A}(X)} = G^{\mathcal{A}(X)}\setminus\{0\}$. $\Box$

We consider some properties associated with involutive semigroups. Suppose now that G has a continuous involution $s \mapsto s^*$, i.e. $(s^*)^* = s$ and $(st)^* = t^*s^*$. For $f \in \mathcal{CB}(G)$ we define $f^*(s) = \overline{f(s^*)}$. A left [right] homogeneous subspace X of $\mathcal{CB}(G)$ will be called *involutive* if it is closed under the involution $f \mapsto f^*$ and the involution is isometric on X. In this case if $M \in X^*$ (here X^* still denotes the dual space), then we define $M^* \in X^*$ by $M^*(f) = \overline{M(f^*)}$.

Proposition 2.10. *Suppose G admits a continuous involution and X is a left introverted involutive homogeneous subspace of $\mathcal{CB}(G)$.*

(i) *The map $M \mapsto M^*$ is a conjugate-linear, weak*-weak* continuous, isometric involution on X^*.*

(ii) *The space X is introverted. On X^* we have $(M\Box N)^* = N^*\Diamond M^*$ and G^X is $*$-closed.*

(iii) *The algebras $\mathcal{E}(X)$, $\mathcal{E}_1(X)$, $\mathcal{A}(X)$ and $\mathcal{A}_1(X)$ are involutive and $\Phi_{\mathcal{A}(X)}$ is $*$-closed.*

Proof. The proof of (i) is straightforward. We prove (ii). Let $f \in X$ and $M \in X^*$. Then for s,t in S we have $(s\cdot f)^*(t) = \overline{s\cdot f(t^*)} = \overline{f(t^*s)} = f^*(s^*t) = f^*\cdot s^*(t)$ so

$$f\cdot M(s) = M(s\cdot f) = \overline{M^*(f^*\cdot s^*)} = \overline{M^*\cdot f^*(s^*)} = (M^*\cdot f^*)^*(s)$$

and hence X is also right introverted, hence introverted. Now it follows for M,N in X^* and f in X that

$$(M\Box N)^*(f) = \overline{(M\Box N)(f^*)} = \overline{M(N\cdot f^*)} = M^*(f\cdot N^*) = N^*\Diamond M^*(f).$$

Finally, if $\chi \in G^X$, then $\chi = \lim_\alpha \varepsilon_X(s_\alpha)$ for some net $(s_\alpha) \subset G$, so we have for f in X

$$\chi^*(f) = \overline{\chi(f^*)} = \lim_\alpha \overline{f^*(s_\alpha)} = \lim_\alpha f(s_\alpha^*)$$

so $\chi^* = \text{weak*-}\lim_\alpha \varepsilon_X(s_\alpha^*) \in G^X$ too.

We prove (iii). If p is a polynomial in n variables with $p(0) = 0$, let $\bar{p}$ denote that same polynomial with conjugated coefficients. Now if $f_1,\dots,f_n \in X$ [or are in $X\cup\bar{X}$], then $p(f_1,\dots,f_n)^* = \bar{p}(f_1^*,\dots,f_n^*)$ remains in $\mathrm{alg}(X)$ [respectively $\mathrm{alg}(X+\bar{X})$]. Hence it follows from continuity of the involution that $\mathcal{A}(X)$ and

$\mathcal{E}(X)$ are involutive. Clearly $1^* = 1$ so $\mathcal{E}_1(X)$ and $\mathcal{A}_1(X)$ are involutive too. If $\chi \in \Phi_{\mathcal{A}(X)}$ then for f, g in $\mathcal{A}(X)$, $\chi^*(fg) = \overline{\chi(f^*g^*)} = \overline{\chi(f^*)\chi(g^*)} = \chi^*(f)\chi^*(g)$. Clearly $\chi^* \neq 0$ since $\chi \neq 0$, so $\chi^* \in \Phi_{\mathcal{A}(X)}$ too. □

2.3. **On semi-topological compactifications.** We specialise our analysis above to semi-topological compactifications. It is well known that (δ, S) is a semitopological compactification of G if and only if (δ, S) is a factor of the weakly almost periodic compactification $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$ associated to the weakly almost periodic functions $\mathcal{WAP}(G)$; or, equivalently, if and only if $\mathcal{C}(S)\circ\delta \subset \mathcal{WAP}(G)$; see [6, §4.2], for example.

We summarise and build upon results due mainly to Glicksberg [25] following Grothendieck [28] to prove a well-known result; see [6, 4.2.7], for example. We reprove this to demonstrate how these properties amount to little more than properties of convolutions of measures.

Theorem 2.11. **(i)** *Let (δ, S) be a semitopological compactification of G. Then $\mathcal{C}(S)\circ\delta$ is introverted and on $(\mathcal{C}(S)\circ\delta)^* \cong \mathrm{M}(S)$ the left and right Arens products coincide and are given by convolution:*

$$\mu{*}\nu(f) = \iint_{S\times S} f(\chi'\chi)d\nu(\chi)d\mu(\chi') = \iint_{S\times S} f(\chi'\chi)d\mu(\chi')d\nu(\chi) \tag{2.6}$$

for $\mu, \nu \in \mathrm{M}(S)$ and f in $\mathcal{C}(S)$

(ii) *Let $\mathcal{X}$ be a closed, translation invariant subspace of $\mathcal{WAP}(G)$. Then $\mathcal{X}$ is introverted and Arens regular.*

Proof. Let $\mu, \nu \in \mathrm{M}(S) \cong \mathcal{C}(S)^*$ and $f \in \mathcal{C}(S)$. Then for s in G we have

$$\nu\cdot(f\circ\delta)(s) = \nu((f\circ\delta)\cdot s) = \int_S f(\delta(s)\chi)d\nu(\chi)$$

and, similarly, $(f\circ\delta)\cdot\mu(s) = \int_S f(\chi'\delta(s))d\mu(\chi')$. Thanks to [25, 1.2] we have that $\nu\cdot f$ and $f\cdot\mu$ given by

$$\nu\cdot f(\chi) = \int_S f(\chi\chi')d\nu(\chi') \text{ and } f\cdot\mu(\chi) = \int_S f(\chi'\chi)d\mu(\chi')$$

are elements of $\mathcal{C}(S)$, hence $\mathcal{C}(S)\circ\delta$ is introverted. Thus, the Fubini theorem [25, 3.1] shows that

$$\begin{aligned}\mu\square\nu(f\circ\delta) = \mu(\nu\cdot(f\circ\delta)) &= \iint_{S\times S} f(\chi'\chi)d\nu(\chi)d\mu(\chi') \\ &= \iint_{S\times S} f(\chi'\chi)d\mu(\chi')d\nu(\chi) = \nu((f\circ\delta)\cdot\mu) = \mu\Diamond\nu(f\circ\delta)\end{aligned}$$

Hence left and right Arens products coincide on measures as functionals on $\mathcal{C}(S)$. Thus (i) is established.

We prove (ii). By Theorem 2.4 (ii), $G^{\mathcal{X}} \cong G^{\mathcal{E}_1(\mathcal{X})}$. Since $\mathcal{E}_1(\mathcal{X}) \subset \mathcal{WAP}(G)$, it follows Theorem 2.1 that $S = G^{\mathcal{X}}$ is a semitopological semigroup. We have that $\mathcal{X} = \mathcal{F}\circ\varepsilon_{\mathcal{X}}$ for some closed $\delta(G)$-translation invariant subspace $\mathcal{F}$ of $\mathcal{C}(S)$. We first note that $\mathcal{F}$ is also S-translation invariant. Given s in S let (t_α) be a net from G so $s = \lim_\alpha \delta(t_\alpha)$. Hence for s' in S we have $s\cdot f(s') = f(s's) = \lim_\alpha f(s'\delta(t_\alpha)) = \lim_\alpha \delta(t_\alpha)\cdot f(s)$, so $s\cdot f = \text{pointwise-}\lim_\alpha \delta(t_\alpha)\cdot f$. By [28, Theo. 5], also see [6, A.2], $s\cdot f = \text{weak-}\lim_\alpha \delta(t_\alpha)\cdot f$. Hence it follows from the Hahn-Banach theorem that $s\cdot f \in \mathcal{F}$. A symmetric argument gives that $f\cdot s \in \mathcal{F}$ too.

Now, let $f \in \mathcal{F}$, $m \in \mathcal{F}^*$ and let μ in $\mathrm{M}(S)$ be so $\mu|_{\mathcal{F}} = m$. We note that $m\cdot f = \mu\cdot f$. We let $\mu = \text{weak*-}\lim_\alpha \mu_\alpha$, where each μ_α is a finite linear combination of point masses of norm not exceeding $\|\mu\|_1$, which may be realised with aid of the Krein-Milman theorem. Since $\mathcal{F}$ is translation invariant, $\mu_\alpha\cdot f \in \mathcal{X}$ for each α, and hence for $s \in S$ we have $\mu\cdot f(s) = \mu(f\cdot s) = \lim_\alpha \mu_\alpha(f\cdot s) = \lim_\alpha \mu_\alpha\cdot f(s)$. Thus $\mu\cdot f = \text{pointwise-}\lim_\alpha \mu_\alpha\cdot f$, and, as above, we deduce that $\mu\cdot f \in \mathcal{F}$. By a symmetric argument we have that $f\cdot\mu \in \mathcal{F}$ too. It follows that $\mathcal{X} = \mathcal{F}\circ\varepsilon_{\mathcal{X}}$ is introverted. Finally, since $\mu \mapsto \mu|_{\mathcal{F}}$ is a quotient homomorphism by Proposition 2.2 (iii), Arens regularity of $\mathcal{C}(S)$ passes to that of $\mathcal{F}$, and hence to $\mathcal{X}$. □

Corollary 2.12. *If X is a homogeneous subspace of $\mathcal{WAP}(G)$ then*

(i) *G^X is a semitopological semigroup;*

(ii) *$\mathcal{E}(X)$ and $\mathcal{A}(X)$ are introverted;*

(iii) *the Arens product on $\mathcal{E}(X)^* \cong \mathrm{M}(G^X\setminus\{0\})$ is given by convolution product: if $0 \in G^X$ we let for f in $\mathcal{C}_0(G^X\setminus\{0\})$ and μ,ν in $\mathrm{M}(G^X\setminus\{0\})$*

$$\mu*\nu(f\circ\varepsilon_X) = \int_{G^X\setminus\{0\}} f(\chi)d(\mu*\nu)(\chi) = \iint_{G^X\times G^X} \tilde{f}(\chi\chi')d\tilde{\mu}(\chi)d\tilde{\nu}(\chi') \tag{2.7}$$

where $\tilde{f}$ is the continuous extension of f to G^X satisfying $\tilde{f}(0) = 0$, and $\tilde{\mu},\tilde{\nu}$ in $\mathrm{M}(G^X)$ are any measures which restrict on Borel subsets of $G^X\setminus\{0\}$ to μ and ν; and

(iv) *$\Phi_{\mathcal{A}(X)}\cup\{0\}$ is a semitoplogical semigroup under convolution product.*

Proof. It follows from the proof of Theorem 2.4 that $\mathcal{E}_1(X)$ is a translation invariant unital C*-subalgebra of $\mathcal{WAP}(G)$, and hence $G^{\mathcal{E}_1(X)} \cong G^X$ is a semi-topological semigroup. Thus we obtain (i). Part (ii) is immediate from (ii) of the theorem above.

By Theorem 2.4, the convolution formula (2.7) of part (iii) extends (2.6) only when $1 \notin \mathcal{E}(X)$. In this case $f \mapsto \tilde{f} : \mathcal{C}_0(G^X\setminus\{0\}) \to \mathcal{C}(G^X)$ is the canonical embedding and we essentially appeal to Proposition 2.2 (iii). We remark that one can select $\tilde{\mu}$ and $\tilde{\nu}$, in (2.7) to satisfy $\tilde{\mu}(\{0\}) = 0 = \tilde{\nu}(\{0\})$.

Finally (iv) is immediate from Proposition 2.8 and the theorem above. □

It may fail that X, above is itself introverted.

Example 2.13. **(i)** Let $G = \mathrm{F}_\infty$, the free group on countably many generators, and consider the space $X = \mathrm{A}^{cb}(G)$, the norm-closure of $\mathrm{A}(G)$ in the completely bounded multipliers $\mathrm{M}_{cb}\mathrm{A}(G)$. See [13] for the definition of $\mathrm{M}_{cb}\mathrm{A}(G)$ and its isometric predual $\mathrm{Q}(G)$, and [22] for more on $\mathrm{A}^{cb}(G)$. It is noted in [62] that $\mathrm{M}_{cb}\mathrm{A}(G) \subset \mathcal{WAP}(G)$. If $\mathrm{A}^{cb}(G)$ were itself introverted then by Proposition 2.2, its dual space $\mathrm{A}^{cb}(G)^*$ would be a Banach subalgebra contained in $\mathrm{VN}(G) \cong \mathrm{A}(G)^*$ containing the algebra $\lambda_1(\ell^1(G))$. The weak amenability property of G (see [13]) tells us that $\mathrm{A}^{cb}(G)$ is weak*-dense in $\mathrm{M}_{cb}\mathrm{A}(G)$, hence the adjoint of the inclusion map $\mathrm{A}^{cb}(G) \hookrightarrow \mathrm{M}_{cb}\mathrm{A}(G)$ takes $\mathrm{Q}(G)$ isometrically onto the closed subspace generated $\lambda_1(\ell^1(G))$ in $\mathrm{A}^{cb}(G)^*$. However computations of Haagerup [29] (see [53, Remark 3.2]) show that $\mathrm{Q}(G)$ is not a Banach algebra with respect to this product. Thus it cannot be the case that $\mathrm{A}^{cb}(G)$ is introverted.

(ii) Let G be any infinite discrete group. Then $\ell^1(G)$ is a homogeneous subspace of $\mathcal{C}_0(G) \subset \mathcal{WAP}(G)$, but is not introverted. Indeed the constant function 1 in $\ell^\infty(G)$ satisfies $1\cdot f(s) = \sum_{t\in G} f(t)$ for each s, i.e. is constant, and is generally not an element of $\ell^1(G)$.

If G is an involutive semigroup, a right [left] topological compactification (δ, S) of G is called *involutive* if there is a continuous involution $x \mapsto x^*$ on S such that $\delta(s^*) = \delta(s)^*$ for s in G.

Proposition 2.14. *Suppose G admits a continuous involution.*

(i) *If a right topological semigroup S admits a continuous involution, then it is semitopological.*

(ii) *If (δ, S) is an involutive right topological compactification, then S is semitopological, and $\mathcal{C}(S)\circ\delta$ is involutive in the sense of Proposition 2.10.*

(iii) *The weakly almost periodic compactification $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$ is an involutive compactification.*

In particular, $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$ is the universal involutive compactification of G.

Proof. We have for s in S that $t \mapsto t^*s^*$ is continuous on S, and hence $t \mapsto st = (t^*s^*)^*$ is continuous on S, so $s \in Z_T(S)$. Thus we have (i). Part (ii) is immediate from (i), and the fact that for f in $\mathcal{C}(S)$, $(f\circ\delta)^* = f^*\circ\delta$, where f^* is clearly in $\mathcal{C}(S)$.

To see (iii) we need only show that $\mathcal{WAP}(G)$ is $*$-closed, from which point we may appeal to Proposition 2.10 (i) and (ii). It follows from [6, 4.2.3], $f \in \mathcal{WAP}(G)$ if and only if $G{\cdot}f$, or equivalently $f{\cdot}G$, is relatively weakly compact in $\mathcal{CB}(G)$. Since $(s{\cdot}f^*)^* = f{\cdot}s^*$ for s in G, it follows that $G{\cdot}f^*$ is relatively weakly compact exactly when $f{\cdot}G$ is.

Since $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$ is universal amongst all semi-topological compactifications, it follows from (i) that it dominates any involutive compactification. Hence $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$ is the universal involutive compactification. □

The following should be compared to results in [18].

Corollary 2.15. *Suppose G admits a continuous involution and X is a left introverted homogeneous subspace of $\mathcal{CB}(G)$. Then G^X admits an involution by which (ε_X, G^X) is an involutive compactification if and only if $X \subset \mathcal{WAP}(G)$ and $\mathcal{E}(X)$ is involutive in the sense of Proposition 2.10.*

Curiously, we need not assume that X itself is involutive in the sense of Proposition 2.10, since properties of G^X are determined by the structure of $\mathcal{E}(X)$. Moreover, it does not appear to be the case that having $\mathcal{E}(X)$ involutive necessarily implies that X itself must be, though we have no examples to suggest otherwise.

Proof. We have $\mathcal{E}_1(X) = \mathcal{C}(G^X)\circ\varepsilon_X$ by Theorem 2.1 (i) and Theorem 2.4 (ii). It then follows Proposition 2.14 and Theorem 2.1 (ii) that if (ε_X, G^X) is involutive then $\mathcal{E}_1(X)$, and hence X, is contained in $\mathcal{WAP}(G)$. Also $\mathcal{E}_1(X)$ is clearly involutive; since $1^* = 1$, necessarily $\mathcal{E}(X)$ is involutive too.

Conversely, if $X \subset \mathcal{WAP}(G)$ and $\mathcal{E}(X)$ is involutive, then $\mathcal{E}_1(X)$ is involutive and contained in $\mathcal{WAP}(G)$. Hence by Proposition 2.10 (ii), and Theorem 2.1 (ii) (ε_X, G^X) is involutive as well. □

Example 2.16. We note that if G is a group, the space of uniformly continuous bounded functions $\mathcal{UCB}(G)$ is $*$-closed where $s^* = s^{-1}$ for s in G. In general $G^{\mathcal{UCB}} = G^{\mathcal{UCB}(G)}$ is not an involutive semigroup in our sense, i.e. the involution is not continuous. We note that G has the small invariant neighbourhood [SIN] property if and only if $\mathcal{UCB}(G) = \mathcal{LUC}(G)$; see [31, (4.14)(g)], for example. Thus if G is a locally compact non-compact [SIN]-group then $\mathcal{UBC}(G)$ is not Arens regular, since in this case neither is $\mathcal{LUC}(G)$; see [42] or [51] for example.

2.4. **Compactifications which are semigroups of contractions on Hilbert spaces.** If $\mathcal{H}$ is a Hilbert space, let $\mathcal{B}(\mathcal{H})_{\|\cdot\|\leq 1}$ denote the weak* (weak operator) compact semitopological semigroup of linear contractions on $\mathcal{H}$. A semigroup of Hilbertian contractions is any subsemigroup $S \subset \mathcal{B}(\mathcal{H})_{\|\cdot\|\leq 1}$. We say

(CH) a compactification (δ, S) of G has property (CH) if S is isomorphic to a weak*-closed semigroup of Hilbertian contractions.

We see that there is a universal (CH)-compactification.

Theorem 2.17. *There is a (CH)-compactification $(\varepsilon_{\mathcal{CH}}, G^{\mathcal{CH}})$ which is universal in the sense that every (CH)-compactification of G is a factor of $(\varepsilon_{\mathcal{CH}}, G^{\mathcal{CH}})$.*

Proof. Thanks to [6, 3.3.4] it suffices to verify that the class of (CH) compactifications is closed under subdirect products. If $\{(\delta_i, S_i)\}_{i\in I}$ is a set of (CH) compactifications, where each S_i is isomorphic to a weak*-closed subsemigroup of $\mathcal{B}(\mathcal{H}_i)_{\|\cdot\|\leq 1}$, then $P = \prod_{i\in I} S_i$ is isomorphic to a weak*-closed subsemigroup of $\prod_{i\in I} \mathcal{B}(\mathcal{H}_i)_{\|\cdot\|\leq 1} \tilde{\subset} \mathcal{B}(\ell^2\text{-}\bigoplus_{i\in I} \mathcal{H}_i)_{\|\cdot\|\leq 1}$. Thus if $\delta : G \to P$ is given by $\delta(s) = (\delta_i(s))_{i\in I}$, and $S = \overline{\delta(G)}^{w^*}$, then (δ, S) is the subdirect product $\bigvee_{i\in I}(\delta_i, S_i)$ of this system, in the sense of [6, 3.2.5]. Clearly (δ, S) is a (CH) compactification. □

As a consequence of Theorem 2.1, we see that if $\mathcal{CH}(G) = \mathcal{C}(G^{\mathcal{CH}})\circ\varepsilon_{\mathcal{CH}}$, then $\mathcal{CH}(G) \supset \mathcal{C}(S)\circ\delta$ for every (CH)-compactification (δ, S) of G. What is not clear is whether or not the class of (CH)-compactifications is closed under homomorphism. Thus we call a compactification (δ, S) of G an (FCH)-compactification if it is a factor of a (CH)-compactification, hence a factor of $(\varepsilon_{\mathcal{CH}}, G^{\mathcal{CH}})$. It is immediate from Theorem 2.1 (ii) that (δ, S) is an (FCH)-compactification if and only if $\mathcal{C}(S)\circ\delta \subset \mathcal{CH}(G)$.

Let us see that each concrete (CH) compactification is associated with a particular operator algebra and an associated homogeneous Banach space in $\mathcal{CB}(G)$. This is modeled closely on [2, (2.2)].

Theorem 2.18. *Let $\delta : G \to (\mathcal{B}(\mathcal{H})_{\|\cdot\|\leq 1}, w^*)$ be a continuous homomorphism and $\mathrm{OA}_\delta = \overline{\mathrm{span}\delta(G)}^{w^*}$. Then there is an introverted homogeneous Banach space A_δ in $\mathcal{CB}(G)$ such that $\mathrm{A}_\delta^* \cong \mathrm{OA}_\delta$. Moreover OA_δ is a subalgebra of $\mathcal{B}(\mathcal{H})$, and the Arens products on OA_δ coincide and are exactly the operator products.*

Proof. The unique predual of $\mathcal{B}(\mathcal{H})$ is given by the projective tensor product $\mathcal{H}\widehat{\otimes}\bar{\mathcal{H}}$, via the identification $T(\xi \otimes \bar{\eta}) = \langle T\xi|\eta\rangle$. Define $E_\delta : \mathcal{H}\widehat{\otimes}\bar{\mathcal{H}} \to \mathcal{CB}(G)$ by

$$E_\delta x(s) = \sum_{j=1}^{\infty} \langle \delta(s)\xi_j|\eta_j\rangle$$

for $x = \sum_{j=1}^{\infty} \xi_j \otimes \bar{\eta}_j$ in $\mathcal{H}\widehat{\otimes}\bar{\mathcal{H}}$ and s in G. If we let $\mathrm{A}_\delta = \mathrm{ran}E_\delta$ be given the norm by which E_δ is a quotient map, then A_δ is a Banach space and we have for $\varepsilon > 0$ and x as above with $\sum_{j=1}^{\infty} \|\xi_j\| \|\eta_j\| < \|x\| + \varepsilon < \|E_\delta x\| + 2\varepsilon$ that for s in G

$$|E_\delta x(s)| \leq \sum_{j=1}^{\infty} |\langle \delta(s)\xi_j|\eta_j\rangle| \leq \|E_\delta x\| + 2\varepsilon$$

so $\|E_\delta x\|_\infty \leq \|E_\delta x\|$, and A_δ is indeed a subspace of $\mathcal{CB}(G)$. The bipolar theorem shows that $\mathrm{OA}_\delta = (\ker E_\delta)^\perp$ and hence it follows from the Hahn-Banach theorem

that $\mathrm{A}_\delta^* \cong (\ker E_\delta)^\perp = \mathrm{OA}_\delta$. The duality relation may be realised by $N(\langle\delta(\cdot)\xi|\eta\rangle) = \langle N\xi|\eta\rangle$ for N in OA_δ and ξ, η in $\mathcal{H}$.

We verify that A_δ is introverted. First note that for ξ, η in $\mathcal{H}$ we have

$$\langle\delta(\cdot)\xi_j|\eta_j\rangle\cdot s = \langle\delta(\cdot)\delta(s)\xi_j|\eta_j\rangle \text{ and } s\cdot\langle\delta(\cdot)\xi_j|\eta_j\rangle = \langle\delta(\cdot)\xi_j|\delta(s)^*\eta_j\rangle$$

from which it follows that A_δ is translation invariant. Now, for N in OA_δ and $\xi, \eta \in \mathcal{H}$ we have that

$$N\cdot\langle\delta(\cdot)\xi|\eta\rangle(s) = N(\langle\delta(\cdot)\xi|\delta(s)^*\eta\rangle) = \langle\delta(s)N\xi|\eta\rangle.$$

Hence, if N in OA_δ, and x in $\mathcal{H}\widehat{\otimes}\bar{\mathcal{H}}$ as above, we have

$$\|N\cdot(E_\delta x)\| = \left\|\sum_{j=1}^\infty \langle\delta(\cdot)N\xi_j|\eta_j\rangle\right\| \leq \sum_{j=1}^\infty \|N\xi_j\|\,\|\eta_j\| \leq \|N\|\,(\|E_\delta x\| + 2\varepsilon).$$

A similar calculation establishes that $\|(E_\delta x)\cdot N\| \leq \|N\|\,(\|E_\delta x\| + 2\varepsilon)$, hence it follows that A_δ is introverted. In particular, setting $N = \delta(s)$ we see that A_δ is homogeneous as well.

It is immediate that, OA_δ, being a weak*-closure of a subalgebra of the dual Banach algebra $\mathcal{B}(\mathcal{H})$, is itself closed under operator multiplication. Let us check the left Arens product. If $M, N \in \mathrm{OA}_\delta$ and ξ, η in $\mathcal{H}$ we have

$$M\Box N(\langle\delta(\cdot)\xi|\eta\rangle = M(N\cdot\langle\pi(\cdot)\xi|\eta\rangle) = M(\langle\delta(\cdot)N\xi|\eta\rangle) = \langle MN\xi|\eta\rangle.$$

The computations for the right Arens product are similar. $\square$

We note that if G is an involutive semigroup and $\delta : G \to (\mathcal{B}(\mathcal{H})_{\|\cdot\|\leq 1}, w^*)$ is a continuous $*$-homomorphism, then OA_δ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, in fact a von Neumann algebra if δ is non-degenerate. Also, A_δ is involutive in the sense of Section 2.2, by a calculation similar to that for (1.1). We give a mild refinement of Proposition 2.2. We maintain the convention of Section 2.2 of stating results only for left introverted spaces and right topological semigroups.

Proposition 2.19. *Let $\delta : G \to (\mathcal{B}(\mathcal{H})_{\|\cdot\|\leq 1}, w^*)$ be a continuous homomorphism, and X a left introverted homogeneous Banach space of $\mathcal{CB}(G)$. Then $\mathrm{A}_\delta \subset X$ boundedly (contractively) if and only if there is a weak*-weak* continuous (contractive) homomorphism $\delta_X : X^* \to \mathrm{OA}_\delta$ for which $\delta_X(\varepsilon_X(s)) = \delta(s)$ for s in G. Moreover, if G is an involutive semigroup, δ is a $*$-homomorphism, and X is involutive, then $\delta_X(M^*) = \delta_X(M)^*$ for M in X^*.*

Proof. The first statement is immediate from Proposition 2.2 (iii) and Theorem 2.18.

Further if X is involutive and δ is a $*$-homomorphism, then for $M \in X^*$ we have for ξ, η in $\mathcal{H}$, using (1.1), that

$$\langle\delta_X(M^*)\xi|\eta\rangle = \overline{M(\langle\delta(\cdot)\eta|\xi\rangle)} = \langle\delta_X(M)^*\xi|\eta\rangle.$$

Notice that δ_X is a homomorphism with respect to either Arens product and $\delta_X(M\Box N) = \delta_X(M\diamond N)$. This is to be expected as elements of A_δ are weakly almost periodic functions. $\square$

We shall see in Corollary 3.15 that there are compact semitopological involutive semigroups G for which $G^{\mathcal{CH}} \neq G$. Hence for such semigroups no analogue of the Gelfand-Raikov theorem (as stated in [31, (22.12)], for example) holds.

3. Eberlein compactifications

From this point forward, let G denote a locally compact group.

3.1. **Basic theory.** If $\pi \in \Sigma_G$ we let $G^\pi = \overline{\pi(G)}^{w^*} \subset \mathrm{VN}_\pi \cong \mathrm{A}_\pi^*$. The pair

$$(\pi, G^\pi)$$

is called the π-*Eberlein compactification* of G. Moreover we set

$$(\varepsilon_{\mathcal{E}}, G^{\mathcal{E}}) = (\omega_G, G^{\omega_G})$$

and simply call this the *Eberlein compactification* of G. The representations τ_π and ρ_π and spaces $\mathrm{A}_\pi, \mathcal{E}_\pi, \mathrm{A}(\pi), \mathcal{A}(\pi), \mathrm{E}(\pi)$ and $\mathcal{E}(\pi)$ are defined in Section 1.3. We remark that in the notation of Section 2.2 we have

$$\mathcal{A}(\pi) = \mathcal{A}(X),\ \mathcal{E}(\pi) = \mathcal{E}(X) \quad \text{and} \quad \mathcal{E}_1(\pi) = \mathcal{E}_1(X)$$

where $X = \mathrm{A}_\pi$; in particular $(\pi, G^\pi) = (\varepsilon_X, G^X)$.

Proposition 3.1. *If $\pi \in \Sigma_G$, then (π, G^π) is an involutive semitopological compactification of G. We have an equivalence of compactifications*

$$(\pi, G^\pi) \cong (\tau_\pi, G^{\tau_\pi}) \cong (\rho_\pi, G^{\rho_\pi}).$$

Moreover, G^π is the Gelfand spectrum of the π-Eberlein algebra $\mathcal{E}_1(\pi)$, and $G^\pi \setminus \{0\}$ is that of $\mathcal{E}(\pi)$.

Proof. That (π, G^π) is an involutive semitopological compactification can be verified directly as involution of VN_π is weak*-weak* continuous. However this can also be deduced from results above. Indeed, on $X = \mathrm{A}_\pi$ we note that $f^* = \bar{\check{f}}$, so X is involutive by (1.1). Thus X^* admits a linear involution by Proposition 2.10. Finally, $X \subset \mathcal{WAP}(G)$, by [7, Theo. 3.11] or [15, Theo. 11.3], for example, so the involution on G^X is a continuous semigroup involution by Corollary 2.15.

We have that $(\pi, G^\pi) \cong (\varepsilon_{\mathcal{E}_1(\pi)}, \Phi_{\mathcal{E}_1(\pi)})$ by Theorem 2.4; and from there we also identify the spectrum of $\mathcal{E}(\pi)$. Since $\mathcal{E}_1(\pi) = \mathcal{E}_1(X)$ for either of $X = \mathrm{A}(\pi), \mathrm{E}(\pi)$, the equivalence of each of (τ_π, G^{τ_π}) and (ρ_π, G^{ρ_π}) with $(\pi, G^\pi) \cong (\varepsilon_X, G^X)$ holds by Corollary 2.5. $\square$

Let us review some established methods of comparing representations π, σ in Σ_G. We say

- π is *quasi-contained* in σ, $\pi \leq_q \sigma$, if $F_\pi \subset \mathrm{A}_\sigma$; and
- π is *weakly contained* in σ, $\pi \leq_w \sigma$, if $F_\pi \subset \mathrm{B}_\sigma$.

We add two more useful comparisons:

- π is *m-quasi-contained* in σ, $\pi \leq_{mq} \sigma$, if $F_\pi \subset \mathrm{A}(\sigma)$; and
- π is *$\bar{m}$-quasi-contained* in σ, $\pi \leq_{\bar{m}q} \sigma$, if $F_\pi \subset \mathrm{E}(\sigma)$.

Using our spaces we define notions of "Eberlein containment":

- π is *strongly Eberlein contained* in σ, $\pi \leq_{se} \sigma$, if $F_\pi \subset \mathcal{E}_\sigma$;
- π is *m-Eberlein contained* in σ, $\pi \leq_{me} \sigma$, if $F_\pi \subset \mathcal{A}(\sigma)$;
- π is *Eberlein contained* in σ, $\pi \leq_e \sigma$, if $F_\pi \subset \mathcal{E}(\sigma)$; and
- π is *weakly Eberlein contained* in σ, $\pi \leq_{we} \sigma$, if $F_\pi \subset \mathcal{EB}(\sigma)$,

where $\mathcal{EB}(\pi)$ is defined in Section 1.3. Each of the relations above is transitive with the exception of weak Eberlein containment; see Example 3.3 (ii), below. For each comparison $\pi \leq_\bullet \sigma$, above, we obtain an equivalence $\pi \cong_\bullet \sigma$ in the case that $\sigma \leq_\bullet \pi$ holds, as well.

The following gives the motivation for including conjugation and multiplication properties in our definition of Eberlein containment. It is immediate from Corollary 2.5 and Proposition 3.1.

Proposition 3.2. *If $\pi, \sigma \in \Sigma_G$ we have*
(i) $\pi \leq_e \sigma \oplus 1$ *if and only if* $(\pi, G^\pi) \leq (\sigma, G^\sigma)$*, and*
(ii) $\pi \oplus 1 \cong_e \sigma \oplus 1$ *if and only if* $(\pi, G^\pi) \cong (\sigma, G^\sigma)$*.*

For π, σ in Σ_G our definitions provide the following implications.

$$\begin{tikzcd}
\pi \leq_q \sigma \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & & \pi \leq_{mq} \sigma \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \pi \leq_{\bar{m}q} \sigma \arrow[d, Rightarrow, "(1)"'] \\
& \pi \leq_{se} \sigma \arrow[r, Rightarrow] & \pi \leq_{me} \sigma \arrow[r, Rightarrow] & \pi \leq_e \sigma \arrow[d, Rightarrow] \\
\pi \leq_w \sigma \arrow[rrr, Rightarrow, "(2)"] & & & \pi \leq_{we} \sigma
\end{tikzcd} \tag{3.1}$$

We shall note, after Theorem 3.11, that this diagram simplifies significantly if π is finite dimensional. None of the implications above are equivalences, in general, showing that these notions of containment are distinct. We will show this, for implications (1) and (2), by way of some examples.

Example 3.3. For any non-compact G, $\mathrm{B}_0(G) = \mathrm{B}(G) \cap \mathcal{C}_0(G)$ is a closed, translation invariant, conjugate invariant subalgebra of $\mathrm{B}(G)$, called the Rajchman algebra. Hence, by [2, (3.17)] there is a representation ρ_0 for which $\mathrm{B}_0(G) = \mathrm{A}_{\rho_0} = \mathrm{E}(\rho_0)$. In general

$$\lambda \leq_q \rho_0, \text{ in particular } \lambda \leq_{\bar{m}q} \rho_0. \tag{3.2}$$

The well known fact that $\mathcal{E}_\lambda = \mathcal{C}_0(G)$ gives $\lambda \cong_e \rho_0$, and, moreover that $(\lambda, G^\lambda) \cong (\rho_0, G^{\rho_0})$ is the one-point compactification.

(i) If G is abelian, then neither of the converse quasi-containments to (3.2) hold; see [26, 7.4.1], for example. This shows that the converse of (1) fails.

(ii) If $G = \mathrm{SL}_2(\mathbb{R})$, then for any non-trivial complementary series representation κ_s $(0 < s < 1)$, we have $\kappa_s \leq_q \rho_0$ by [34, V.2.0.3], for example, so $\kappa \leq_e \rho_0 \cong_e \lambda$. However it follows from Harish-Chandra's trace formula [30] that $\kappa_s \not\leq_w \lambda$ for any s. Hence the converse to (2) fails; in particular, Eberlein containment can hold in a situation where weak containment fails.

Now, consider the representation $\kappa = \bigoplus_{0<s<1} \kappa_s$. As indicated above, $\kappa \leq_e \lambda$, and, since $\lambda \cong \bar{\lambda}$ it follows that $\kappa \leq_{we} \lambda$. However it follows from [49] (see [21, §7.6]) that $1 \leq_w \kappa$, so $1 \leq_{we} \kappa$. However, G is non-amenable so $1 \not\leq_w \lambda$ (we may also use Harish-Chandra's trace formula [30] to see this). Again by [34, V.2.0.3], this implies that $\mathrm{B}_\lambda \subset \mathcal{C}_0(G)$, and it follows that $1 \not\leq_{we} \lambda$.

We remark that by [33, Theo. 6.1], for any algebraic group G over a local field and any non-trivial irreducible representation π we have that $\pi \leq_e \rho_0$.

Recall that the definition of an (η, H)-regular compactification was given before Proposition 2.7.

Theorem 3.4. *A right topological compactification (δ, S) of G is such that S has an open group of units, if and only if (δ, S) is a regular (η, H)-compactification for some locally compact right topological group H with proper right translations. In particular, if $\mathcal{C}(S) \circ \delta \supset \mathcal{C}_0(G)$, then $\delta(G) \cong G$ is the open group of units in S.*

Proof. If the group H of units of S is open, then H itself is a locally compact right toplogical group. Hence (δ, S) is (δ, H)-regular. If, on the other hand, (δ, S) is (η, H)-regular, then by (2.5) in Proposition 2.7, the group of units is isomorphic to H, and hence open.

By Proposition 2.7, (δ, S) is (id, G)-regular exactly when $\mathcal{C}(S)\circ\delta \supset \mathcal{C}_0(G)$. $\square$

We note that not every Eberlein compactification has an open group of units. See Example 4.3 (iii). Fortunately $G^{\mathcal{E}}$ has an open group of units.

Corollary 3.5. *Let G and H be locally compact groups, $\pi \in \Sigma_G$ satisfy $\lambda_G \leq_e \pi$, and $\sigma \in \Sigma_H$ satisfy $\lambda_H \leq_e \sigma$. Then if $G^\pi \cong H^\sigma$ as semitopological semigroups, we must have $G \cong H$. In particular, $G^{\mathcal{E}} \cong H^{\mathcal{E}}$ if and only if $G \cong H$.*

Proof. In light of Proposition 3.2, our assumptions imply that $\mathcal{C}_0(G) = \mathcal{E}(\lambda_G) \subset \mathcal{E}(\pi) + \mathbb{C}1$. Thus $\mathcal{C}_0(G) \subset \mathcal{E}(\pi)$, and the theorem above tells us that the group of units of G^π is isomorphic to G. The result follows. $\square$

We observe that we can repeat the arguments above to see that if $q : G \to G/\ker\pi$ is the quotient map, and $\lambda_{G/\ker\pi}\circ q \leq_e \pi$, then the group of units of G^π is isomorphic to $G/\ker\pi$.

3.2. **Subalgebras generated by matrix coefficients.** We pass to the case of the algebras $\mathcal{A}(\pi)$ shortly, with the aid of a straightforward generalisation of [63, Theo. 1]. Below we let $\mathrm{VN}(\pi) = \mathrm{VN}_{\tau_\pi}$. Also, if $\sigma \leq_q \pi$, we let $\sigma^\pi : \mathrm{VN}_\pi \to \mathrm{VN}_\sigma$ be the normal $*$-representation given by taking the adjoint of the inclusion map $\mathrm{A}_\sigma \hookrightarrow \mathrm{A}_\pi$; see, for example, Proposition 2.19, above. Note that this notation satisfies $\sigma^\pi(\pi(s)) = \sigma(s)$ for s in G.

Theorem 3.6. *Let $\pi \in \Sigma_G$ and $x \in \mathrm{VN}(\pi) \cong \mathrm{A}(\pi)^*$. Then the following are equivalent*

(i) $x \in \Phi_{\mathrm{A}(\pi)} \cup \{0\}$*;*
(ii) $(\sigma\otimes\rho)^{\tau_\pi}(x) = \sigma^{\tau_\pi}(x) \otimes \rho^{\tau_\pi}(x)$ *for any* $\sigma, \rho \leq_{mq} \pi$*; and*
(iii) $(\pi\otimes\pi)^{\tau_\pi}(x) = \pi^{\tau_\pi}(x) \otimes \pi^{\tau_\pi}(x)$.

Thus $\Phi_{\mathrm{A}(\pi)}\cup\{0\}$ is a $*$-semigroup in $\mathrm{VN}(\pi)$, which may be isomorphically identified with the $*$-semigroup $\pi^{\tau_\pi}(\Phi_{\mathrm{A}(\pi)}) \cup \{0\}$ in VN_π. Also, if $x \in \pi^{\tau_\pi}(\Phi_{\mathrm{A}(\pi)})$ has polar decomposition $x = v|x|$, then $v, |x| \in \pi^{\tau_\pi}(\Phi_{\mathrm{A}(\pi)})$ too.

If $\pi\otimes\pi \leq_q \pi$, i.e. $\tau_\pi \cong_q \pi$, then for $x \in \mathrm{VN}_\pi$ we have

$$x \in \Phi_{\mathrm{A}_\pi} \cup \{0\} \quad \textit{if and only if} \quad (\pi\otimes\pi)^\pi(x) = x \otimes x. \tag{3.3}$$

We shall indicate in Example 4.2, below, that $\Phi_{\mathrm{A}(\pi)}$ is not itself a semigroup, in general.

Proof. We first note that for typical elements of $\mathrm{A}(\pi)$, say $\langle\sigma(\cdot)\xi|\eta\rangle$ and $\langle\rho(\cdot)\vartheta|\zeta\rangle$ where $\sigma, \rho \leq_{mq} \pi$, we have $\langle\sigma\otimes\rho(\cdot)\xi\otimes\vartheta|\eta\otimes\zeta\rangle = \langle\sigma(\cdot)\xi|\eta\rangle\,\langle\rho(\cdot)\vartheta|\zeta\rangle$. Thus we have that $x \in \Phi_{\mathrm{A}(\pi)} \cup \{0\}$ if and only if

$$\langle(\sigma\otimes\rho)^{\tau_\pi}(x)\xi\otimes\vartheta|\eta\otimes\zeta\rangle = \langle\sigma^{\tau_\pi}(x)\xi|\eta\rangle\,\langle\rho^{\tau_\pi}(x)\vartheta|\zeta\rangle$$

where the latter expression is simply $\langle\sigma^{\tau_\pi}(x)\otimes\rho^{\tau_\pi}(x)\xi\otimes\vartheta|\eta\otimes\zeta\rangle$. Hence we have the equivalence of (i) and (ii). That (ii) implies (iii) is clear. That (iii) implies (i) follows from the computation above, since F_π generates $\mathrm{A}(\pi)$. By the same fact we see that π^{τ_π} must be injective on $\Phi_{\mathrm{A}(\pi)} \cup \{0\}$. The closure of $\Phi_{\mathrm{A}(\pi)}$ under polar decomposition follows exactly as in [64, Theo. 1], hence this happens in $\pi^{\tau_\pi}(\Phi_{\mathrm{A}(\pi)})$ too.

We note that $\pi\otimes\pi\leq_q\pi$ implies that A_π is itself an algebra, in which case $\tau_\pi\leq_q\pi$, and hence $\tau_\pi\cong_q\pi$. In this case π^{τ_π} is an isomorphism with inverse $(\tau_\pi)^\pi$, and (3.3) is immediate from the equivalence of (i) and (ii) applied to elements $(\tau_\pi)^\pi(x)$. □

If $\sigma\leq_{se}\pi$, Proposition 2.19 provides a $*$-homomorphism $\sigma_e^\pi:\mathcal{E}_\pi^*\to\mathrm{VN}_\sigma$. We note, moreover, that if $\sigma\leq_e\pi$ then $\sigma_e^{\rho_\pi}:\mathcal{E}(\pi)^*\cong\mathrm{M}(G^\pi\setminus\{0\})\to\mathrm{VN}_\sigma$ may be realised by the integral formula

$$\langle\sigma_e^\pi(\mu)\xi|\eta\rangle=\int_{G^\pi\setminus\{0\}}\langle\theta(x)\xi|\eta\rangle\,d\mu(x)$$

where $\theta:G^\pi\to G^\sigma$ is the factor map. Also, if $\sigma,\rho\leq_{me}\pi$, then $\sigma\otimes\rho\leq_{me}\pi$, and hence we can define $(\sigma\otimes\rho)_e^{\tau_\pi}:\mathcal{A}(\pi)^*\to\mathrm{VN}_{\sigma\otimes\rho}$. Finally, since A_π is dense in $\mathcal{E}_\pi$, $\pi_e^\pi:\mathcal{E}_\pi^*\to\mathrm{VN}_\pi$ is injective, and hence we can identify $\mathcal{E}_\pi^*$ as a linear subspace of VN_π

Corollary 3.7. *Let $\pi\in\Sigma_G$ and $\mu\in\mathcal{A}(\pi)^*$. Then the following are equivalent*

(i) $\mu\in\Phi_{\mathcal{A}(\pi)}\cup\{0\}$

(ii) $(\sigma\otimes\rho)_e^{\tau_\pi}(\mu)=\sigma_e^{\tau_\pi}(\mu)\otimes\rho_e^{\tau_\pi}(\mu)$ *for any* $\sigma,\rho\leq_{me}\pi$*; and*

(iii) $(\pi\otimes\pi)_e^{\tau_\pi}(\mu)=\pi_e^{\tau_\pi}(\mu)\otimes\pi_e^{\tau_\pi}(\mu)$.

Moreover, $\pi_e^{\tau_\pi}:\Phi_{\mathcal{A}(\pi)}\cup\{0\}\to\pi_e^{\tau_\pi}(\Phi_{\mathcal{A}(\pi)}\cup\{0\})$ *is a $*$-isomorphism and homeomorphism, in particular latter set is a $*$-semigroup in* VN_π *which contains* G^π*.*

If $\pi\cong_{se}\bar{\pi}$ we can replace (i) by

(i') $\mu\in G^{\mathcal{A}(\pi)}$ *(in which case $\pi_e^{\tau_\pi}(\mu)=x$ for some x in $G^\pi\setminus\{0\}$, or $\mu=0$) and $\leq_e$ can replace $\leq_{me}$ in (ii).*

Proof. Since $\mathrm{A}(\pi)$ is dense in $\mathcal{A}(\pi)$, the computations of Theorem 3.6 can be repeated. That $\Phi_{\mathcal{A}(\pi)}\cup\{0\}$ is a semigroup on which $\pi_e^{\tau_\pi}$ is an isomorphism follows from the facts above, or alternatively, Theorem 2.9 (i). If $\pi\cong_{se}\bar{\pi}$, then (i') follows from the standard fact that the spectrum of $\mathcal{A}(\pi)=\mathcal{E}(\pi)\cong\mathcal{C}_0(G^\pi\setminus\{0\})\cong G^\pi\setminus\{0\}$, and, of course, 0 is allowed in the case that $1\not\leq_e\pi$. □

We gain an augmentation of Corollary 3.5.

Proposition 3.8. **(i)** *If $\pi\in\Sigma_G$ satisfies $\lambda\leq_{mq}\pi$, then the semigroup $\Phi_{\mathrm{A}(\pi)}\cup\{0\}$ has an open group of units, isomorphic to G.*

(ii) *Suppose G and H are locally compact groups and π in Σ_G and $\sigma\in\Sigma_H$ are such that $\lambda_G\leq_{mq}\pi$ and $\lambda_H\leq_{mq}\sigma$. If the semigroups $\Phi_{\mathrm{A}(\pi)}\cup\{0\}$ and $\Phi_{\mathrm{A}(\sigma)}\cup\{0\}$ are continuously isomorphic, then $G\cong H$.*

Proof. We first prove (i). We first observe that $\varepsilon_{\mathrm{A}(\pi)}(G)$ is open in $\Phi_{\mathrm{A}(\pi)}$. Indeed, $\varepsilon_{\mathrm{A}(\pi)}:G\to\Phi_{\mathrm{A}(\pi)}$ is continuous, and injective, since $\mathrm{A}(\pi)\supset\mathrm{A}(G)$. We note that if $x\in\Phi_{\mathrm{A}(\pi)}$ and $\langle u,x\rangle\neq0$ for some u in $\mathrm{A}(G)$, then $x\in\varepsilon_{\mathrm{A}(\pi)}(G)$; indeed, there is s in G for which $\langle w,x\rangle=w(s)$ for w in $\mathrm{A}(G)$ by [17, (3.34)], and since $\mathrm{A}(G)$ is an ideal in $\mathrm{A}(\pi)$ — i.e. an ideal in $\mathrm{B}(G)$ — it follows that $x=\varepsilon_{\mathrm{A}(\pi)}(s)$. Now, if $s\in G$, by [17, (3.2)], find compactly supported v in $\mathrm{A}(G)$ so that $v(s)=1$. If $(x_i)\subset\Phi_{\mathrm{A}(\pi)}$ is a net converging to $\varepsilon_{\mathrm{A}(\pi)}(s)$ then $\lim_i\langle v,x_i\rangle=v(s)=1$, so eventually $\langle v,x_i\rangle\neq0$. Then, for such i, $x_i=\varepsilon_{\mathrm{A}(\pi)}(s_i)$ for some s_i in $U=\{s\in G:v(s)\neq0\}$. Thus $\varepsilon_{\mathrm{A}(\pi)}(U)$ is a neighbourhood of $\varepsilon_{\mathrm{A}(\pi)}(s)$, from which it follows that $\varepsilon_{\mathrm{A}(\pi)}(G)$ is open in $\Phi_{\mathrm{A}(\pi)}$.

Next we note that a simple modification of the proof of (2.5) in Proposition 2.7, along with the characterisation of $\varepsilon_{\mathrm{A}(\pi)}(G)$ in $\Phi_{\mathrm{A}(\pi)}$, above gives that $\Phi_{\mathrm{A}(\pi)}\setminus$

$\varepsilon_{\mathrm{A}(\pi)}(G)$ is an ideal in $\Phi_{\mathrm{A}(\pi)} \cup \{0\}$. Thus we see that $\varepsilon_{\mathrm{A}(\pi)}(G) \cong G$ is the subgroup of the identity $\tau_\pi(e)$, in $\Phi_{\mathrm{A}(\pi)} \cup \{0\}$, and is open.

The result (ii) follows immediately from (i). $\square$

It is not generally true that $\Phi_{\mathcal{A}(\pi)} = \Phi_{\mathrm{A}(\pi)}$. We call $\Phi_{\mathrm{A}(\pi)} \setminus \Phi_{\mathcal{A}(\pi)}$ the *Wiener-Pitt part* of $\Phi_{\mathrm{A}(\pi)}$. If $\Phi_{\mathrm{A}(\pi)} = \Phi_{\mathcal{A}(\pi)}$ we will say that π is *spectrally natural*; if not we will say it is a *Wiener-Pitt* representation. This nomenclature is motivated by the example of Wiener and Pitt, exhibiting a non-neagitive element of $\mathrm{B}(\mathbb{Z})$ whose spectrum caontains i; see [39, VII 9.1]. Whenever G admits a non-compact abelian quotient $H \cong G/N$, then G admits Wiener-Pitt representations. For example consider $\omega_H \circ q$ or even $\rho_0 \circ q$ (ρ_0 is the Rajchman representation of H defined in Example 3.3) where $q : G \to H$ is the quotient map; see [56, 6.4.1] or [26, 8.2.3] for example. On the other hand, if G is a connected semi-simple Lie group with finite centre, then it follows from [11] that ω_G is spectally natural.

Part (ii) of the following, if true, would be useful to us in Section 4.

Conjecture 3.9. **(i)** *If $\sigma \leq_q \pi$ and π is spectrally natural, then σ is spectrally natural.*

(ii) *If $\sigma \leq_q \lambda$, where λ is the left regular representation, then σ is spectrally natural.*

We observe that if $\sigma \leq_q \lambda$ then $\sigma \leq_{\bar{m}q} \lambda$ as well. It follows [4, Theo. 2.1] that $\mathrm{E}(\sigma) \cong \mathrm{A}(G/\ker\sigma)$ and $\ker\sigma$ is a compact subgroup of G. Hence $\mathrm{A}(\sigma)$ is a closed translation-invariant subalgebra of $\mathrm{E}(\sigma) \cong \mathrm{A}(G/\ker\sigma)$, and thus $\mathcal{A}(\sigma)$ is a closed translation invariant subalgebra of $\mathcal{C}_0(G/\ker\sigma)$. This may help in resolving (ii).

We end this section by relating for $\pi \in \Sigma_G$, the algebra $\mathcal{A}(\pi)$ to the π-Eberlein compactification G^π.

Theorem 3.10. *If $\pi \in \Sigma_G$, then $\partial_{\mathcal{A}(\pi)} = G^{\mathcal{A}(\pi)} \setminus \{0\}$.*

Proof. We shall find it convenient to make the identification $G^{\mathcal{A}(\pi)} \cong G^{\tau_\pi}$, which may be facilitated by applying Corollary 3.7 to τ_π instead of to π itself. By Theorem 2.9 we have that $G^{\tau_\pi} \supset \partial_{\mathcal{A}(\pi)}$, and it suffices to show that $\partial_{\mathcal{A}(\pi)} \cap \tau_\pi(G) \neq \varnothing$ to see that $\partial_{\mathcal{A}(\pi)} = G^{\tau_\pi} \setminus \{0\}$.

For $\xi \in \mathcal{H}_{\tau_\pi}$ with $\|\xi\| = 1$ we let

$$u_\xi(s) = \frac{1}{2}\langle \tau_\pi(s)\xi|\xi\rangle + \frac{1}{2}\langle \tau_\pi(s)\xi|\xi\rangle^2$$

so u_ξ is a positive definite element of $\mathcal{A}(\pi)$ with $\|u_\xi\|_\infty = u_\xi(e) = 1$. We have for $x \in G^{\tau_\pi}$ that $|\langle x\xi|\xi\rangle| \leq 1$ and hence

$$1 = |\langle u_\xi, x\rangle| = \frac{1}{2}|\langle x\xi|\xi\rangle + \langle x\xi|\xi\rangle^2| \quad \Leftrightarrow \quad \langle x\xi|\xi\rangle = 1.$$

Hence $K_\xi = \{x \in G^{\tau_\pi} : \langle x\xi|\xi\rangle = 1\}$ is the maximal norming set for u_ξ, i.e. K_ξ is the set of elements x in G^{τ_π} for which $|\langle u_\xi, x\rangle| = 1$.

Now suppose $B \subset G^{\tau_\pi}$ is a boundary for $\mathcal{A}(\pi)$. Note that $B \cap K_\xi$ is compact for every ξ, and non-empty by considerations above. For each finite collection $\xi_1, \ldots, \xi_n$ of norm one vectors we have that $\bigcap_{j=1}^n K_{\xi_j}$ is maximally norming for $u_{\xi_1} + \cdots + u_{\xi_n}$ and hence $\bigcap_{j=1}^n (B \cap K_{\xi_j})$ is non-empty. By the finite intersection property $\bigcap\{B \cap K_\xi : \xi \in \mathcal{H}_{\tau_\pi}, \|\xi\| = 1\} \neq \varnothing$. However, $\bigcap\{K_\xi : \xi \in \mathcal{H}_{\tau_\pi}, \|\xi\| = 1\} = \{\tau_\pi(e)\}$, thus $\tau_\pi(e) \in B$. Hence $\tau_\pi(e) \in \partial_{\mathcal{A}(\pi)}$. $\square$

We remark that if $\lambda \leq_{me} \pi$, i.e. $\mathcal{C}_0(G) \subset \mathcal{A}(\pi)$, then each $\pi(s)$, s in G, is a strong boundary point, and the theorem above is trivial.

3.3. **Eberlein weak containment and amenability.** Let us briefly review the position of almost periodic functions within $\mathcal{E}(G)$. This approach is a variant of the one taken in [57, §2]. Let $\mathcal{AP}(G)$ denote the C*-algebra of almost periodic functions on G, and $(\varepsilon_{\mathcal{AP}}, G^{\mathcal{AP}})$ denote the almost periodic compactification. Let $\hat{G}_F$ denote the collection of finite dimensional irreducible representations in Σ_G, and $\pi_F = \bigoplus_{\sigma\in\widehat{G}_F}\sigma$. Then $\mathrm{A}_F(G) = \mathrm{A}_{\pi_F} = \mathrm{A}(G^{\mathcal{AP}})\circ\varepsilon_{\mathcal{AP}} = \mathrm{B}(G) \cap \mathcal{AP}(G)$ (see [17, (2.20)]), and hence $\mathcal{E}(\pi_F) = \mathcal{AP}(G)$. A straightforward adaptation of [7, Thm. 2.22] shows that there is a minimal idempotent $e_{\mathcal{AP}}$ in the semitopological semigroup $G^{\mathcal{E}}$ for which $\mathcal{AP}(G) = \mathcal{C}(G^{\mathcal{AP}})\circ\varepsilon_{\mathcal{AP}} = e_{\mathcal{AP}}\cdot\mathcal{E}(G)$. Combining comments above we see that $\mathrm{A}_F(G) = \mathrm{A}(G^{\mathcal{AP}})\circ\varepsilon_{\mathcal{AP}} = e_{\mathcal{AP}}\cdot\mathrm{B}(G)$, qua subspaces of $\mathcal{E}(G)$. Moreover, $f \mapsto e_{\mathcal{AP}}\cdot f$ is a $*$-homomorphism since $e_{\mathcal{AP}}$ is the minimal idempotent in $G^{\mathcal{E}} = \Phi_{\mathcal{E}(G)}$. With the identifications in Corollary 3.7 applied to $\pi = \omega_G$, we see that $e_{\mathcal{AP}}$ in $G^{\mathcal{AP}}$ corresponds to the central projection p_F in $\mathrm{W}^*(G)$ which covers π_F. (The role of p_F is discussed in [64], where it is denoted z_F, but it is not discussed whether this projection is in $G^{\mathcal{E}}$.) In particular we obtain closed ideals

$$\mathrm{A}_{PI}(G) = \{f \in \mathrm{B}(G) : p_F\cdot f = 0\} \text{ and } \mathcal{E}_0(G) = \{f \in \mathcal{E}(G) : e_{\mathcal{AP}}\cdot f = 0\}$$

of $\mathrm{B}(G)$ and $\mathcal{E}(G)$, respectively. Here $\mathrm{A}_{PI}(G)$ stands for the "purely infinite" part of $\mathrm{B}(G)$; this is conjugation closed and translation invariant since it is $\mathrm{B}(G)\cap\mathcal{E}_0(G)$, and hence by [2, (3.17)] there is a representation π_{PI} for which $\mathrm{A}_{PI}(G) = \mathrm{A}_{\pi_{PI}}$. By [2, (3.12) & (3.14)], $\pi \leq_q \pi_{PI}$ if and only if π contains no finite dimensional subrepresentations. We note that $\mathcal{E}_0(G)$ is exactly the space of functions in $\mathcal{E}(G)$ whose absolute values are in the kernel of the invariant mean on $\mathcal{WAP}(G)$; this can be adapted from [7, Cor. 2.18]. These ideals give rise to "semi-direct product" decompositions

$$\mathrm{B}(G) = \mathrm{A}_{PI}(G) \oplus_{\ell^1} \mathrm{A}_F(G) \text{ and } \mathcal{E}(G) = \mathcal{E}_0(G) \oplus \mathcal{AP}(G).$$

We may regard the following as a refinement of Theorem 2.4 (iii).

Theorem 3.11. *If $\pi \in \Sigma_G$, then the following are equivalent:*
(i) $\sigma \leq_q \pi$ *for some* σ *in* $\widehat{G}_F$*;*
(i') $\sigma \leq_{se} \pi$ *for some* σ *in* $\widehat{G}_F$*;*
(ii) $1 \leq_{\bar{m}q} \pi$*;*
(ii') $1 \leq_e \pi$*;*
(iii) $\mathcal{E}_\pi \cap \mathcal{AP}(G) \neq \{0\}$*;*
(iii') $\mathcal{E}(\pi) \cap \mathcal{AP}(G) \neq \{0\}$*; and*
(iv) $0 \notin G^\pi$*.*

We remark that this holds for $\pi = \lambda$ if and only if G is compact.

Proof. That (i) implies (i'), and (ii) implies (ii') are obvious. If (i) holds then $1 \leq_q \sigma \otimes \bar{\sigma} \leq_{\bar{m}q} \pi$, so (ii) holds. Similarly (i') implies (ii'). Condition (ii') implies that $1 \in \mathcal{E}(\pi) \cap \mathcal{AP}(G)$, and hence we obtain (iii'). Theorem 2.4 (iii) gives the equivalence of (ii') and (iv). It remains to prove that (iii) implies (i), and (iii') implies (iii).

If it were the case that $p_F \cdot \mathrm{A}_\pi = \{0\}$, then for any u in $\mathcal{E}_\pi$ and any sequence $(u_n) \subset \mathrm{A}_\pi$ for which $\lim_{n\to\infty} \|u - u_n\|_\infty = 0$, we would have $e_{\mathcal{AP}}\cdot u = \lim_{n\to\infty} e_{\mathcal{AP}}\cdot u_n = \lim_{n\to\infty} p_F \cdot u_n = 0$, thus $e_{\mathcal{AP}} \cdot \mathcal{E}_\pi = \{0\}$. If (iii) holds then $e_{\mathcal{AP}} \cdot \mathcal{E}_\pi =$

$\mathcal{E}_\pi \cap \mathcal{AP}(G) \neq \{0\}$, and hence $\mathrm{A}_\pi \cap \mathrm{A}_F(G) = p_F{\cdot}\mathrm{A}_\pi \neq \{0\}$, whence there is ρ in Σ_G for which $\mathrm{A}_\pi \cap \mathrm{A}_F(G) = \mathrm{A}_\rho$. Since every ρ for which $\mathrm{A}_\rho \subset \mathrm{A}_F(G)$ is completely reducible, we see that statement (i) follows from (iii).

If it were the case that (iii) did not hold, i.e. $\mathcal{E}_\pi \cap \mathcal{AP}(G) = \{0\}$, then we would have that $\mathrm{A}_\pi \cap \mathrm{A}_F(G) = \{0\}$. Hence by [2, (3.12)], π would contain no subrepresentation of π_F, in which case $\pi \leq_q \pi_{PI}$, and hence $\mathrm{A}_\pi \subset \mathrm{A}_{PI}(G)$. Since $\mathrm{A}_{PI}(G)$ is a closed and conjugation-closed subalgebra, in fact ideal, of $\mathrm{B}(G)$, it would follow that $\mathrm{A}(\pi) \cap \mathrm{A}_F(G) = \{0\}$. In this case, we would have by the same argument of the the above paragraph that $e_{\mathcal{AP}}{\cdot}\mathcal{E}(\pi) = \{0\}$, hence we would obtain $\mathcal{E}(\pi) \cap \mathcal{AP}(G) = \{0\}$, which would violate (iii'). Thus (iii') implies (iii). □

As a consequence of the above theorem, we observe that for a finite dimensional, hence totally reducible, representation π, and any σ in Σ_G, that $\pi \leq_q \sigma$ if and only if $\pi \leq_{se} \sigma$. Hence the diagram (3.1) may be simplified by noting that each implication between the first and second rows is an equivalence.

Following [3, Theo. 5.1], we say that π in Σ_G is *amenable* if and only if $1 \leq_w \pi\otimes\bar{\pi}$.

Theorem 3.12. *Let $\pi \in \Sigma_G$. Then the following are equivalent:*

(i) $1 \leq_{we} \pi$*;*

(ii) $\sigma \leq_{we} \pi$ *for some* σ *in* $\widehat{G}_F$*; and*

(iii) $\mathcal{EB}(\pi) \cap \mathcal{AP}(G) \neq \{0\}$.

Moreover, each of the above conditions implies each of the following conditions

(iv) π *is amenable; and*

(v) $\mathcal{EB}(\pi \otimes \bar{\pi}) \cap \mathcal{AP}(G) \neq \{0\}$*;*

which are equivalent to one another; and each of conditions (i)-(iii) *is implied by* (iv) *or* (v)*, provided that* $\pi \otimes \bar{\pi} \leq_w \pi$

Proof. First, that (i) implies (ii) is clear. If (ii) holds, then $1 \leq \sigma \otimes \bar{\sigma}$, so $1 \in F_{\sigma\otimes\bar{\sigma}} \subset \mathrm{alg}(F_\sigma + F_{\bar{\sigma}})$, whence we get (iii).

Fell continuity of conjugation and (1.1) tell us that $\mathrm{B}_{\bar{\pi}} = \bar{\mathrm{B}}_\pi$, i.e. $\omega_{\bar{\pi}} = \bar{\omega}_\pi$. Hence condition (i) is the same as saying $1 \leq_e \omega_\pi$, while (iii) is the same as saying $\mathcal{E}(\omega_\pi) \cap \mathcal{AP}(G) \neq \{0\}$. Hence the equivalence of (i) and (iii) follows from the equivalence of (ii') and (iii') of Theorem 3.11, above. Moreover, we see by the equivalence of (i) and (ii') of Theorem 3.11, that condition (i) of the present theorem implies that $\sigma \leq_q \omega_\pi$ for some σ in $\widehat{G}_F$. Hence $1 \leq \sigma\otimes\bar{\sigma} \leq_q \omega_\pi\otimes\omega_{\bar{\pi}} \leq_q \omega_{\pi\otimes\bar{\pi}}$. The last containment holds since $\mathrm{B}_\pi\mathrm{B}_{\bar{\pi}} \subset \mathrm{B}_{\pi\otimes\bar{\pi}}$ by virtue of the facts that $F_\pi F_{\bar{\pi}} = F_{\pi\otimes\bar{\pi}}$ and multiplication is weak*-continuous on $\mathrm{B}(G)$. Hence we see (i) implies (iv).

If (v) holds, then by the equivalence of (i) and (iii') of Theorem 3.11, $\sigma \leq_q \omega_{\pi\otimes\bar{\pi}}$ for some σ in $\widehat{G}_F$. Hence it follows that $\pi \otimes \bar{\pi}$ is amenable, and therefore π is amenable (see [3, Theo. 1.3] and [58, Prop. 2.7]). Hence we get (iv). Conversely, if (iv) holds, then $1 \in \mathrm{B}_{\pi\otimes\bar{\pi}} \subset \mathcal{EB}(\pi \otimes \bar{\pi}) \cap \mathcal{AP}(G)$ so (v) holds.

If we assume that $\pi \otimes \bar{\pi} \leq_w \pi$, and (iv) holds, then $1 \leq_w \pi \otimes \bar{\pi} \leq_w \pi$. Thus (i) holds. □

We remark that there are no evident containment relations between $\mathcal{EB}(\pi)$ and $\mathcal{EB}(\pi \otimes \bar{\pi})$, in general. However, if $1 \leq_w \pi$ we have $\mathcal{EB}(\pi) \subset \mathcal{EB}(\pi \otimes \bar{\pi})$.

We now define the *reduced Eberlein algebra* and *reduced Eberlein compactification* by

$$\mathcal{E}_r(G) = \overline{\mathrm{B}_r(G)}^{\|\cdot\|_\infty} \text{ and } (\varepsilon_{\mathcal{E}_r}, G^{\mathcal{E}_r}) = (\varepsilon_{\mathcal{E}_r(G)}, G^{\mathcal{E}_r(G)})$$

where $\mathrm{B}_r(G) = \mathrm{B}_\lambda$ and $\lambda : G \to \mathcal{U}(\mathrm{L}^2(G))$ is the left regular representation. The following is an analogue of Hulanicki's theorem [35] — *G is amenable if and only if each continuous positive definite function can be approximated, uniformly on compacta, by positive definite functions associated with the left regular representation* — which is equivalent to saying that $\omega_G \leq_w \lambda$, or that $1 \leq_w \lambda$. However, parts (viii), (ix) and (x) have no obvious analogue in that context.

Theorem 3.13. *The following are equivalent:*
(i) *G is amenable;*
(ii) *$\mathcal{E}_r(G) = \mathcal{E}(G)$;*
(iii) *$1 \in \mathcal{E}_r(G)$;*
(iv) *$\mathrm{M}(G^{\mathcal{E}_r} \setminus \{0\}) \cong \mathcal{E}_r(G)^*$ and $\mathrm{M}(G^{\mathcal{E}}) \cong \mathcal{E}(G)^*$ are isomorphic dual Banach algebras;*
(v) *$\mathrm{M}(G^{\mathcal{E}_r} \setminus \{0\}) \cong \mathcal{E}_r(G)^*$ admits a weak* continuous character;*
(vi) *$\sigma \leq_{we} \lambda$ for some $\sigma \in \hat{G}_F$; and*
(vii) *$\pi \leq_{we} \lambda$ for every π in Σ_G.*
If we further assume that $\hat{G}_F \setminus \{1\} \neq \varnothing$, then (i)-(vii), above, are equivalent to
(viii) *$(\sigma, G^\sigma) \leq (\varepsilon_{\mathcal{E}_r}, G^{\mathcal{E}_r})$ for some σ in $\hat{G}_F \setminus \{1\}$;*
(ix) *$(\pi, G^\pi) \leq (\varepsilon_{\mathcal{E}_r}, G^{\mathcal{E}_r})$ for every π in Σ_G; and*
(x) *$(\varepsilon_{\mathcal{E}_r}, G^{\mathcal{E}_r}) \cong (\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$.*

Proof. If (i) holds, then $\mathrm{B}_r(G) = \mathrm{B}(G)$ and hence (ii) holds. That (ii) implies both (iii) and (iv) is obvious. Condition (iii) is that $1 \leq_{we} \lambda$, which by Theorem 3.12 implies that λ is amenable. Hence by [3, Theo. 2.2] (see also [20, p. 260], since we have used an equivalent definition of "amenable representation") we obtain (i). Since $\pi \leq_{we} \omega_G$ holds generally, and (ii) says that $\omega_G \cong_{we} \lambda$, (ii) and (vii) are immediately equivalent.

If (iii) holds then, by Theorem 2.4, $0 \notin G^{\mathcal{E}_r}$. Moreover $\mu * \nu(1) = \mu(\nu \cdot 1) = \mu(\nu(1)1) = \mu(1)\nu(1)$ for μ, ν in $\mathrm{M}(G^{\mathcal{E}_r}) \cong \mathcal{E}_r(G)^*$, so (v) holds. Similarly, $\mathrm{M}(G^{\mathcal{E}}) \cong \mathcal{E}(G)^*$ always admits a weak* continuous character, so if (iv) holds, then so too must (v). If (v) holds, then there is h in $\mathcal{E}_r(G)$ such that $\mu * \nu(h) = \mu(h)\nu(h)$ for each μ, ν in $\mathrm{M}(G^{\mathcal{E}_r} \setminus \{0\}) \cong \mathcal{E}_r(G)^*$. Thus we have $h(st) = \delta_{st}(h) = \delta_s * \delta_t(h) = \delta_s(h)\delta_t(h) = h(s)h(t)$ for s, t in G. Hence h is a norm one character, so $h \in \widehat{G}_F$ with $h \leq_{we} \lambda$, and we have (vi). Converesly, if (vi) holds then by Theorem 3.12 we obtain $1 \leq_{we} \lambda$ and hence we obtain (iii).

Condition (vii) implies condition (ix), which in turn implies condition (viii). If there is σ in $\hat{G}_F \setminus \{1\}$ satisfying condition (viii), then by virtue of Proposition 3.2 we have that $\mathbb{C}1 \neq \mathcal{E}_\sigma \subset \mathcal{E}_r(G) \oplus \mathbb{C}1$, from which it follows that $\mathcal{AP}(G) \cap \mathcal{E}_r(G) \supset \mathcal{E}_\sigma \cap \mathcal{E}_r(G) \neq \{0\}$. Hence from the equivalence of (iii) and (ii') in Theorem 3.11, $1 \leq_e \omega_\lambda$ which in turn gives (iii) in the present theorem. The choice $\omega = \pi$ in (ix) gives that $(\varepsilon_{\mathcal{E}}, G^{\mathcal{E}}) \leq (\varepsilon_{\mathcal{E}_r}, G^{\mathcal{E}_r})$; whereas the converse comparison holds by Proposition 3.2. Thus (ix) implies (x). That (x) implies (ix) follows from Proposition 3.2. $\square$

If $G = \mathrm{SL}_2(\mathbb{R})$, then $\hat{G}_F = \{1\}$ and $(\varepsilon_{\mathcal{E}}, G^{\mathcal{E}}) = (\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$ is the one-point compactification; see Example 3.3 (ii) and [10]. Hence $(\varepsilon_{\mathcal{E}_r}, G^{\mathcal{E}_r}) = (\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$, in this case, and the condition $(\pi, G^\pi) \leq (\varepsilon_{\mathcal{E}_r}, G^{\mathcal{E}_r})$ of (ix) is satisfied, despite that G is not amenable.

3.4. **Universal property of the Eberlein compactification.** We recall the definition of the (CH)-compactification from Section 2.4. The following theorem appears not to be new, but appears in a dual form, *à la* Theorem 2.1, in [48, Thm. 3.12]. Moreover no assumption of local compactness of the the topological group G is made in [48]. Our proof appears to hold in that setting as well, and is sufficiently different to merit inclusion.

Theorem 3.14. *The Eberlein compactification $(\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$ is the (CH) compactification $(\varepsilon_{\mathcal{CH}}, G^{\mathcal{CH}})$, and hence it is universal amongst (CH) compactifications.*

Proof. We first note that (CH) compactifications are exactly those of the form (π, G^π) where $\pi \in \Sigma_G$. Indeed each such (π, G^π) is clearly a (CH) compactification. Conversely, if $\delta : G \to \mathcal{B}(\mathcal{H})_{\|\cdot\|\leq 1}$ is a homomorphism, then $p = \delta(e)$ is a contractive idempotent, hence a projection. For any s in G let $u = \delta(s)$ and $v = \delta(s^{-1})$. Then $uv = vu = p$. Moreover if $\xi \in p\mathcal{H}$ then $\|\xi\| = \|vu\xi\| \leq \|u\xi\| \leq \|\xi\|$, and if $\xi \in (1-p)\mathcal{H}$ then $u\xi = u(1-p)\xi = 0$, which shows that u is a partial isometry with support and range projection p, and hence $v = u^*$. We note that p commutes with $S = \overline{\delta(G)}^{w^*}$. We let $\mathcal{H}_\pi = p\mathcal{H}$ and $\pi = p\delta(\cdot)|_{p\mathcal{H}}$, so π is a unitary representation. Since $S = pSp$ we see that $x \mapsto px|_{\mathcal{H}_\pi}$ is a continuous bijection from S onto G^π, intertwining δ and π. Hence $(\delta, S) \cong (\pi, G^\pi)$.

It is immediate from Proposition 3.2, and the fact that each $\pi \in \Sigma_G$ satisfies $\pi \leq_q \omega_G$, hence $\pi \leq_e \omega_G$, that each (CH) compactification is a factor of $(\varepsilon_{\mathcal{E}}, G^{\mathcal{E}}) \cong (\omega_G, G^{\omega_G})$. □

Given $\pi \in \Sigma_G$, we call a compactification (δ, S) an (Fπ)-compactification ("factor of π") if $(\delta, S) \leq (\pi, G^\pi)$. In keeping with Theorem 3.14, we have that an (FCH)-compactification is an $(F\omega_G)$-compactification. It would be interesting to know if every (FCH)-compactification of G is itself a (CH)-compactification. It may be necessary to condsider only involutive (FCH)-compactifications.

We can use Theorem 3.14 to give examples of involutive compact semitopological semigroups which cannot be faithfully represented on Hilbert spaces. We gain an extension of [48, Theo. 4.7], where it is observed via [55] that the compact involutive semigroup $\mathbb{Z}^w$ cannot be represented faithfully as contractions on a Hilbert space. We say that G is *totally minimal* if every continuous homomorphism into another topological group has closed range. It is shown in [45, Thm. 2.5] that connected totally minimal groups are precisely those which are inductive limits of groups of the form $R \rtimes N$, where R is a reductive Lie group acting on a nilpotent Lie group N with no non-trivial fixed points.

Corollary 3.15. *Suppose that G is either (a) nilpotent, (b) has an inner automorphism invariant compact neighbourhood of the identity, i.e. is an [IN]-group, or (c) is connected but not totally minimal. Then $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}}) \not\leq (\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$. In particular $(G^{\mathcal{WAP}})^{\mathcal{CH}}$ is a proper quotient of $G^{\mathcal{WAP}}$, in this case.*

Proof. For cases (a) and (b) it is shown in [9], and for case (c) it is shown in [45, Thm. 4.5], that $\mathcal{WAP}(G) \not\subset \mathcal{E}(G)$. Hence that $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}}) \not\leq (\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$ is immediate from Theorem 2.1 (ii). If it were the case that there were an isomorphism $\theta : G^{\mathcal{WAP}} \to (G^{\mathcal{WAP}})^{\mathcal{CH}}$, then $(\theta\circ\varepsilon_{\mathcal{WAP}}, (G^{\mathcal{WAP}})^{\mathcal{CH}})$ would be a (CH)-compactification of G, which would imply that $(\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}}) \leq (\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$, which contradicts results above. □

It is shown in [45, Thm. 4.5] that all connected groups for which $(\varepsilon_{\mathcal{E}}, G^{\mathcal{E}}) \cong (\varepsilon_{\mathcal{WAP}}, G^{\mathcal{WAP}})$ have Eberlein compactifications of the form illustrated in Proposition 4.1, below.

We also have a complementary "minimality property" of Eberlein type compactifications.

Proposition 3.16. *Let (δ, S) be a right topological compactification of G, and $\pi \in \Sigma_G$. Then the following are equivalent:*

(i) *there is a continuous homomorphism $\pi_S : S \to (\mathcal{B}(\mathcal{H}_\pi), w^*)$ such that $\pi_S \circ \delta = \pi$;*

(ii) $(\pi, G^\pi) \leq (\delta, S)$*;*

(iii) $\mathcal{C}(S) \circ \delta \supset \mathrm{A}_\pi$*.*

In particular, $(\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$ is the minimum compactification for which (i), above, holds for all π in Σ_G.

Proof. That (i) implies (ii) is by definition of the ordering of compactifications. Note that since $\mathcal{B}(\mathcal{H}_\pi)_{\|\cdot\| \leq 1}$ is weak* compact, $\pi_S(S) \subset \mathcal{B}(\mathcal{H}_\pi)_{\|\cdot\| \leq 1}$. Since $\mathcal{E}(\mathrm{A}_\pi) = \mathcal{E}(\pi)$, it follows from Theorem 2.1 that (ii) implies (iii). If (iii) holds, a straighforward modification of Proposition 2.19 provides a weak*-weak* continuous map $\delta_S : \mathcal{C}(S)^* \cong (\mathcal{C}(S) \circ \delta)^* \to \mathrm{VN}_\pi$ for which $\delta_S \circ \varepsilon_{\mathcal{C}(S)} \circ \delta = \pi$. Let $\pi_S = \delta_S \circ \varepsilon_S$ and we obtain (i).

Thus if (δ, S) satisfies (i) for all π in Σ_G, then with the choice of $\pi = \omega$ we obtain that $(\delta, S) \geq (\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$. Of course, if $(\delta, S) \geq (\varepsilon_{\mathcal{E}}, G^{\mathcal{E}})$, then (i) holds for all π in Σ_G. □

4. Examples

As in the previous section, we will always let G denote a locally compact group, unless otherwise indicated.

4.1. **Spine type compactifications.** Suppose H is a locally compact group and $\eta : G \to H$ is a continuous homomorphism with dense range. We will call (η, H) a *locally compact completion* of G. Two locally compact completions (η_j, H_j), $j = 1, 2$, have a *mutual quotient* if there are compact normal subgroups $K_j \subset H_j$ for which $H_1/K_1 \cong H_2/K_2$, via a bicontinuous isomorphism θ for which $\theta \circ q_i \circ \eta_i = q_j \circ \eta_j$, where $q_j : H_j \to H_j/K_j$ is the quotient map for each j. We shall define the *subdirect product* of locally compact completions (η_j, H_j), $j = 1, 2$, as the pair $(\eta_1 \times \eta_2, \overline{\{(\eta_1(s), \eta_2(s)) : s \in G\}})$ and denote it by $(\eta_1, H_1) \vee (\eta_2, H_2)$.

The following is adapted from [36]. We let $\lambda_H : H \to \mathrm{L}^2(H)$ denote the left regular representation of H.

Proposition 4.1. *Let $(\eta_j, H_j)_{j \in J}$ be a family of locally compact completions of G, and*

$$\lambda_J = \bigoplus_{j \in J} \lambda_j \text{ where } \lambda_j = \lambda_{H_j} \circ \eta_j.$$

(i) *No two distinct (η_i, H_i) and (η_j, H_j) are mutually quotient if and only if*

$$\mathrm{A}_{\lambda_J} = \ell^1\text{-}\bigoplus_{j \in J} \mathrm{A}(\lambda_j).$$

Moreover, each $\mathrm{A}(\lambda_j) = \mathrm{A}_{\lambda_j} = \mathrm{A}(H_j) \circ \eta_j \cong \mathrm{A}(H_j)$.

(ii) *If for each i,j in J, $(\eta_i,H_i)\vee(\eta_j,H_j)\cong(\eta_k,H_k)$ for some k in J, we write $k=i\vee j$ and $(J,\vee)$ is a semilattice. Further $\mathrm{A}_{\lambda_J}=\mathrm{A}(\lambda_J)$ is an algebra of functions, graded over $(J,\vee)$ in the sense that $\mathrm{A}(\lambda_i)\mathrm{A}(\lambda_j)\subset\mathrm{A}(\lambda_{i\vee j})$. Moreover, $i\vee j=j$ — we write $i\leq j$ — if and only if there is a continuous homomorphism $\eta^j_i:H_j\to H_i$ such that $\eta^j_i\circ\eta_j=\eta_i$.*

(iii) *Suppose both (i) and (ii) hold and also that the semilattice $(J,\vee)$ is complete. Then for any i,j in J either*

(a) *there is an element $i\wedge j$ in J which satisfies $i\wedge j\leq i$, $i\wedge j\leq j$, and for any k in J for which $k\leq i$ and $k\leq j$, then $k\leq i\wedge j$ too; or*

(b) *there is no k in J for which $k\leq j$ and $k\leq i$.*

If (a) always holds for all i,j then $(J,\wedge)$ is a semilattice; otherwise we can adjoin an identity o, so $o\vee j=j$ for each j in J, and $(J\cup\{o\},\wedge)$ is a semilattice. We obtain identifications

$$\Phi_{\mathrm{A}(\lambda_J)}\cong G^{\lambda_J}\setminus\{0\}\cong\bigsqcup_{j\in J}H_j.$$

The semigroup structure on G^{λ_J} is given for s_i in H_i and s_j in H_j by

$$s_is_j=\begin{cases}\eta^i_{i\wedge j}(s_i)\eta^j_{i\wedge j}(s_j) & \text{if (a) holds}\\ 0 & \text{if (b) holds.}\end{cases}\tag{4.1}$$

The topology is given by basic open neighbourhoods of a point s in H_j that are of the form

$$V_j\sqcup\left\{t\in\bigsqcup_{i\in J,i>j}H_i:\eta^i_{i_k}(t)\in W_{i_k}\ \text{if}\ t\in H_i\ \text{for}\ i\geq i_k\right\}$$

where V_j is an open neighbourhood of s in H_j, each $i_k>j$, and each W_{i_k} is a cocompact set in H_{i_k}.

Proof. Part (i) can proved as in [36, §3.3]. Part (ii) can be proved as in [36, §3.2]. Part (iii) is proved as in [36, §4.1-4.3]. We remark on some details given there. We let

$$\mathfrak{HD}(J)=\left\{S\subset J:\begin{matrix}S\text{ is hereditary: }j\in S,i\leq j & \Rightarrow & i\in S;\\ \text{and }S\text{ is directed: }i,j\in S & \Rightarrow & i\vee j\in S\end{matrix}\right\}.$$

For each $s\in\mathfrak{HD}(J)$ we obtain an inverse mapping system $\{H_j,\eta^j_i:j\in S,i\leq j\text{ in }S\}$ which gives rise to a projective limit

$$H_S=\varprojlim_{j\in S}H_j=\left\{(s_j)_{j\in S}\in\prod_{j\in S}H_j:\eta^j_i(s_j)=s_i\text{ if }i\leq j\text{ in }S\right\}.$$

Then the proof of [36, Theo. 4.1] and remarks on [36, p. 285] gives the structure of $G^{\lambda_J}$and semigroup product by

$$G^{\lambda_J}\cong\bigsqcup_{S\in\mathfrak{HD}(J)}G_S,\quad (s_j)_{j\in S_1}(t_j)_{j\in S_2}=\begin{cases}(s_jt_j)_{j\in S_1\cap S_2} & \text{if }S_1\cap S_2\neq\varnothing\\ 0 & \text{if }S_1\cap S_2=\varnothing.\end{cases}$$

The assumption that $(J,\vee)$ is complete allows that each element of $\mathfrak{HD}(J)$ is principal, i.e. of the form $S_j=\{i\in J:i\leq j\}$, in which case $H_{S_j}\cong H_j$. In the event that $S_i\cap S_j\neq\varnothing$ for each i,j, we obtain (a); otherwise (b) holds. In the case that (b) holds, for some i,j, we adjoin o to J so $o\leq j$ for each j in J. We let $\eta^j_o(s)=0$, and we obtain the product (4.1) as in [36, (4.8)].

The description of the topology is immediate from the corrigendum [37] to [36, Theo. 4.2]. Translated into the present terminology a basic open neighbourhood of $s = (s_j)_{j\in S_0} \in H_{S_0} \subset G^{\lambda_J}$ is of the form

$$\left\{ t \in G^{\lambda_J} : \begin{matrix} t = (t_i)_{i\in S} \in H_S \text{ for some } S \supseteq S_j \text{ in } \mathfrak{H}\mathfrak{D}(J) \\ \text{where } t_j \in V_j \text{ and } t_{i_k} \in W_{i_k} \text{ if } i_k \in S \end{matrix} \right\} \tag{4.2}$$

where $j \in S_0$, V_j is an open neighbourhood of s_j, $i_1, \dots, i_k > j$ and W_{i_k} is a cocompact set in H_{i_k} for each k. □

We note that since G is dense in G^{λ_J}, λ_J is spectrally natural in the sense that $\Phi_{\mathcal{A}(\lambda_J)} = \Phi_{\mathrm{A}(\lambda_J)}$. In fact, it can be verified by way of the regularity condition for Fourier algebras [17, (3.2)] that this algebra is regular on G^{λ_J}. We say that the representation λ_J *spectrally regular* in this case.

Example 4.2. Here we show examples satisfying all of the assumptions of Proposition 4.1 for which $G^{\lambda_J} \setminus \{0\} = \Phi_{\mathcal{A}(\lambda_J)}$ is not a semigroup.

(i) Let H be any non-compact locally compact group and $G = H \times H$. Let $(\eta_o, H_o) = (\mathrm{id}, G)$, $H_l = H = H_r$, and $\eta_l(s,t) = s$, $\eta_r(s,t) = t$ for $(s,t) \in G$. Then $J = \{o, l, r\}$, in the notation above, is the flat semilattice given by $o \vee j = o$ for any j in J and $r \vee l = o$. Clearly there is no j in J for which $j \leq l$ and $j \leq r$, so $G^{\lambda_J} \setminus \{0\}$ is not itself a semigroup.

(ii) Let $G = \mathbb{R}^n$. Fix an inner product on G and for any subspace $L \subset G$, let η_L be the orthogonal projection onto L. If $\mathcal{L}$ denotes the set of subspaces, then $L_1 \vee L_2 = L_1 + L_2$ and $L_1 \wedge L_2 = L_1 \cap L_2$. If we consider, for example, $\mathcal{L}_k = \{L \in \mathcal{L} : \dim L \geq k\}$, for $k = 0, \dots, n$, then $G^{\lambda_{\mathcal{L}_k}} \setminus \{0\}$ is a semigroup only if $k = 0, n$.

Example 4.3. Let $\mathbb{Q}$ be the discrete rationals and G be the direct sum group $\mathbb{Q}^{\oplus\infty}$. Let for each n, $\eta_n : G \to \mathbb{R}^n$ be the projection onto the first n coordinates. We note that $(\eta_n, \mathbb{R}^n) \vee (\eta_m, \mathbb{R}^m) = (\eta_{n\vee m}, \mathbb{R}^{n\vee m})$ where $n \vee m = \max\{n, m\}$, and if $m \leq n$ then $\eta^n_m : \mathbb{R}^n \to \mathbb{R}^m$ is the projection onto the first m coordinates. In the notation of the proof of Proposition 4.1, the projective limit $H_\mathbb{N}$ is isomorphic to the direct sum group $\mathbb{R}^{\oplus\infty}$ with inductive limit topology, hence is not locally compact. We note that every element of $\mathfrak{H}\mathfrak{D}(\mathbb{N})$ is principal except for $\mathbb{N}$ itself. Thus we obtain the structure and semigroup product

$$G^{\lambda_\mathbb{N}} \setminus \{0\} \cong \bigsqcup_{n\in\mathbb{N}\cup\{\infty\}} \mathbb{R}^{\oplus n}, \quad (s_j)_{j=1}^n (t_j)_{j=1}^m = (s_j + t_j)_{j=1}^{n\wedge m},\ n, m \in \mathbb{N} \cup \{\infty\}.$$

The basic open neighbourhoods of $s = (s_j)_{j=1}^\infty \in \mathbb{R}^{\oplus\infty}$, as described in (4.2), are

$$\left\{ t = (t_j)_{j=1}^m \in \bigsqcup_{m=n,\dots,\infty} \mathbb{R}^{\oplus m} : (t_j)_{j=1}^n \in V_n \right\}$$

where V_n is an open neighbourhood of $(t_j)_{j=1}^n$ in $\mathbb{R}^{\oplus n}$. The group of units is $\mathbb{R}^\infty$ and is plainly not open in $G^{\lambda_\mathbb{N}}$.

We wish to extend Proposition 4.1 to include subrepresentations of those of the type λ_J.

Proposition 4.4. *Let $\{\pi_j, \mathcal{H}_j\}_{j\in J}$ be a family of representations for which there is a corresponding family (η_j, H_j) of locally compact completions of G for which*

$\pi_j \leq_q \lambda_{H_j}\circ\eta_j$ for each j. Suppose that the assumptions of (i), (ii) and (iii) in Proposition 4.1 hold and further that

$$\mathrm{A}_{\pi_J} = \ell^1\text{-}\bigoplus_{j\in J} \mathrm{A}(\pi_j)$$

is a subalgebra of A_{λ_J}, where $\pi_J = \bigoplus_{j\in J}\tau_{\pi_j}$. Then for each $i \leq j$ in J there is a semigroup homomorphism $\kappa_i^j : \Phi_{\mathrm{A}(\pi_j)} \cup \{0\} \to \Phi_{\mathrm{A}(\pi_i)} \cup \{0\}$ which satisfies $\kappa_i^j\circ\tau_{\pi_j}(s) = \tau_{\pi_i}(s)$ for each s in G and $\kappa_i^j(0) = 0$. Moreover, $\mathrm{A}_{\pi_J} = \mathrm{A}(\pi_J)$ and

$$\Phi_{\mathrm{A}(\pi_J)} \cong \bigsqcup_{j\in J} \Phi_{\mathrm{A}(\pi_j)}.$$

The semigroup structure on $\Phi_{\mathrm{A}(\pi_J)}\cup\{0\}$ is given for x_i in $\Phi_{\mathrm{A}(\pi_i)}\cup\{0\}$ and x_j in $\Phi_{\mathrm{A}(\pi_j)}\cup\{0\}$ by

$$x_i x_j = \begin{cases} \kappa^i_{i\wedge j}(x_i)\kappa^j_{i\wedge j}(x_j) & \text{if (a) of Proposition 4.1 holds} \\ 0 & \text{if (b) of Proposition 4.1 holds.} \end{cases}$$

The topology is given as follows: basic open neighbourhoods of a point x_j in $\Phi_{\mathrm{A}(\pi_j)}$ are of the form

$$V_j \sqcup \left\{ x \in \bigsqcup_{i\in J, i>j} \Phi_{A(\pi_j)} : \kappa^i_{i_k}(x) \in W_{i_k} \text{ if } x \in \Phi_{\mathrm{A}(\pi_i)} \text{ for } i \geq i_k \right\}$$

where $i_1,\dots,i_k > j$, each W_{i_k} is a cocompact set in $\Phi_{\mathrm{A}(\pi_{i_k})}$, and V_j is an open neighbourhood of s_j in $\Phi_{\mathrm{A}(\pi_j)}$.

Proof. Whilst similar to Proposition 4.1, the present result cannot be deduced from the same proof from [36]. Thus we show some of the details.

The assumption that A_{π_J} is an algebra and the assumptions of Proposition 4.1 (i) and (ii) imply that for $i \leq j$ we have

$$\mathrm{A}(\pi_i)\mathrm{A}(\pi_j) \subset \mathrm{A}_{\pi_J} \cap \big(\mathrm{A}(\lambda_i)\mathrm{A}(\lambda_j)\big) \subset \mathrm{A}_{\pi_J} \cap \mathrm{A}(\lambda_j) = \mathrm{A}(\pi_j).$$

If $i \leq j$ and $x \in \Phi_{\mathrm{A}(\pi_j)}$, define $\kappa_i^j(x)$ in $\mathrm{VN}(\pi_i)$ by

$$\langle u_i, \kappa_i^j(x)\rangle = \frac{\langle u_i u_j, x\rangle}{\langle u_j, x\rangle}$$

for u_i in $\mathrm{A}(\pi_i)$ and u_j in $\mathrm{A}(\pi_j)$ for which $\langle u_j, x\rangle \neq 0$. This is independent of the choice of u_j, since if $u_j' \in \mathrm{A}(\pi_j)$ then

$$\langle u_i u_j, x\rangle\langle u_j', x\rangle = \langle u_i u_j u_j', x\rangle = \langle u_i u_j', x\rangle\langle u_j, x\rangle.$$

We see that that $\kappa_i^j(x) \in \Phi_{\mathrm{A}(\pi_i)}\cup\{0\}$: if $u_i, u_i' \in \mathrm{A}(\pi_i)$ and u_j, u_j' are as above and with $\langle u_j, x\rangle\langle u_j', x\rangle \neq 0$, then

$$\langle u_i, \kappa_i^j(x)\rangle\langle u_i', \kappa_i^j(x)\rangle = \frac{\langle u_i u_j, x\rangle}{\langle u_j, x\rangle}\frac{\langle u_i' u_j', x\rangle}{\langle u_j', x\rangle} = \frac{\langle u_i u_i' u_j u_j', x\rangle}{\langle u_j u_j', x\rangle} = \langle u_i u_i', \kappa_i^j(x)\rangle.$$

It is obvious that $\kappa_i^j(0) = 0$ and straightforward to check that $\kappa_i^j\circ\eta_j = \eta_i$ on G.

We wish to show that κ_i^j is multiplicative on $\Phi_{\mathrm{A}(\pi_j)}\cup\{0\}$. First, if x, u_i, u_j are as above we have for s in G that

$$(u_i u_j)\cdot x(s) = \langle s\cdot(u_i u_j), x\rangle = \langle s\cdot u_j\, s\cdot u_i, x\rangle = \langle s\cdot u_j, \kappa_i^j(x)\rangle\langle s\cdot u_i, x\rangle = u_j\cdot\kappa_i^j(x)u_i\cdot x(s).$$

Hence we find for x_1, x_2 in $\Phi_{\mathrm{A}(\pi_j)}$ we have for u_i, u_j as above

$$\langle u_i, \kappa_i^j(x_1x_2)\rangle = \frac{\langle u_iu_j, x_1x_2\rangle}{\langle u_j, x_1x_2\rangle} = \frac{\langle u_j\cdot\kappa_i^j(x_1)u_i\cdot x_1, x_2\rangle}{\langle u_i\cdot x_1, x_2\rangle} = \langle u_i, \kappa_i^j(x_1)\kappa_i^j(x_2)\rangle.$$

Finally, the structure of $\Phi_{\mathrm{A}(\pi_J)}$ and of the multiplication on $\Phi_{\mathrm{A}(\pi_J)} \cup \{0\}$, can now be deduced from the results indicated in the proof of Proposition 4.1. □

4.2. **Abelian groups.** For this section let G denote a locally compact abelian group. We let $\widehat{G}$ denote the dual group and $\mathrm{L}^1(\widehat{G})$ its group algebra. If $U \subset \widehat{G}$ is any open subset, we let $\mathrm{L}^p(U) = \{f \in \mathrm{L}^p(\widehat{G}) : f = 1_U f\}$, for $p = 1, 2, \infty$. We define

$$\pi_U : G \to \mathcal{U}(\mathrm{L}^2(U)) \text{ by } \pi_U(s)f(\chi) = \chi(s)f(\chi) = \hat{s}(\chi)f(\chi).$$

If $U = G$ then $\pi_G \cong \lambda$ via conjugation by the Plancherel unitary, hence $\pi_U \leq_q \lambda$. In fact, if we let $\lambda_U = \int_U^\oplus \chi\, dm(\chi)$ on $L^2(U) = \int_U^\oplus \mathbb{C}_\chi\, dm(\chi)$ where m is the Haar measure on $\widehat{G}$, then $\pi_U \cong \lambda_U$. The Fourier transform gives $\mathrm{A}(G) \cong \mathrm{L}^1(G)$ and restricts to give the identification $A_{\pi_U} \cong \mathrm{L}^1(U)$. If $S = \bigcup_{n=1}^\infty U^n$ is the semigroup generated by U, we have that $\mathrm{L}^1(S)$ is the closed algebra generated by U, hence $\mathrm{A}(\pi_U) = \mathrm{A}_{\pi_S}$, and we thus consider the algebra

$$\mathrm{A}(\lambda_S) = \mathrm{A}(\pi_S) \cong \mathrm{L}^1(S). \tag{4.3}$$

We let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed disc; while this notation differs from that in most complex analysis texts, it is more convenient for our needs. We let for any open semigroup $S \subset \widehat{G}$

$$\widehat{S} = \{\sigma : S \to \mathbb{D} \,|\, \sigma \text{ is continuous and } \sigma(\chi\chi') = \sigma(\chi)\sigma(\chi') \text{ for } \chi, \chi' \text{ in } S \text{ and } \sigma \neq 0\}$$

denote the set of bounded semicharacters, which is itself an involutive semigroup under pointwise multiplication and conjugation.

Proposition 4.5. *If S is an open subsemigroup of $\widehat{G}$, then $\Phi_{\mathrm{A}(\pi_S)} \cong \widehat{S}$.*

Proof. An elegant way to prove this result is to use the identification (4.3) and prove directly that $\widehat{S} \cong \Phi_{\mathrm{L}^1(S)}$ via the identification $\sigma \mapsto \big(f \mapsto \int_S f(\chi)\sigma(\chi)\, dm(\chi)\big)$. The later identification is shown in [1, 4.1]. However, we wish to emphasise how this follows from Theorem 3.6.

We first observe that $\mathrm{VN}(\pi_S) = \{M_\varphi : \varphi \in \mathrm{L}^\infty(S)\}$, where each operator M_φ is given by $M_\varphi f = \varphi f$. Indeed, it is well known that $\mathrm{span}\{\hat{s} : s \in G\}$ is weak* dense in $\mathrm{L}^\infty(\widehat{G})$, and it follows that $\mathrm{span}\{1_S\hat{s} : s \in G\}$ is weak* dense in $\mathrm{L}^\infty(S) = 1_S\mathrm{L}^\infty(\widehat{G})$. Since $\pi_S \otimes \pi_S \leq_q \pi_S$, we have, in the notation of Theorem 3.6, a normal $*$-homomorphism $(\pi_S \otimes \pi_S)^{\pi_S} : \mathrm{VN}_{\pi_s} \to \mathrm{VN}_{\pi_S\otimes\pi_S} \cong \mathrm{VN}_{\pi_S}\overline{\otimes}\mathrm{VN}_{\pi_S}$. Identifying $\mathrm{L}^2(S)\otimes^2\mathrm{L}^2(S) \cong \mathrm{L}^2(S\times S)$ we compute that $\pi_S\otimes\pi_S(s) = M_{\hat{s}\circ\varsigma}$ where $\varsigma : S\times S \to S$ is the multiplication map. Thus for φ in $\mathrm{span}\{1_S\hat{s} : s \in G\}$, $(\pi_S \otimes \pi_S)^{\pi_S}(M_\varphi) = M_{\varphi\circ\varsigma}$, and by weak* continuity this identification extends to all $\varphi \in \mathrm{L}^\infty(S)$. If $\sigma \in \mathrm{L}^\infty(S)$, $M_\sigma \in \Phi_{A(\pi_S)} \cup \{0\}$ if and only if σ has essential range within $\mathbb{D}$ and, by Theorem 3.6, $\sigma\circ\varsigma = \sigma\otimes\sigma$, i.e. $\sigma(\chi\chi') = \sigma(\chi)\sigma(\chi')$ for a.e. χ, χ' in S. We note that for $f \in \mathrm{L}^1(S)$, $\int_S \sigma(\chi')f(\chi^{-1}\chi')\, dm(\chi') = \sigma(\chi)\int_S \sigma(\chi')f(\chi')\, dm(\chi')$, from which it is immediate that σ is continuous. □

The algebras $\mathrm{A}(\pi_S)$ above, are all algebras of generalised analytic functions in the sense of [1].

It is noted in [1] that the decomposition $\mathbb{D}\setminus\{0\}=(0,1]\times\mathbb{T}$ gives a polar decomposition $\sigma(t)=|\sigma(t)|\mathrm{sgn}\sigma(t)$. In light of the identification (4.3), this is a special case of the polar decomposition observed in Theorem 3.6.

For concreteness, we record some particular examples of semicharacter semigroups whose descriptions we have not been able to locate in the literature. For subgroups of vector groups we always use additive notation.

Proposition 4.6. **(i)** *Let S be any subsemigroup of the additive semigroup $\mathbb{Z}^{\geq 0}$. Then*

$$\widehat{S}\cong\begin{cases}\mathbb{D} & \text{if } 0\in S\\ \mathbb{D}\setminus\{0\} & \text{if } 0\notin S\end{cases}$$

where each semicharacter is given for s in S by $\sigma_z(s)=z^{s/d}$ for some z in $\mathbb{D}$, where $d=\gcd(S\setminus\{0\})$.

(ii) *Let S be any open subsemigroup of the vector group $\mathbb{R}^n$. Then there is a linearly independent subset $\{h_1,\dots,h_n\}$ of $\mathbb{R}^N$ and $0\leq l\leq n$ for which $S\subset\bigoplus_{j=1}^{l}\mathbb{R}h_j\oplus\bigoplus_{j=1}^{n-l}\mathbb{R}^{>0}h_j$. Moreover $\widehat{S}\cong\mathbb{R}^l\times\mathbb{H}^{n-l}$ where $\mathbb{H}=\{z\in\mathbb{C}:\mathrm{Im}z\geq 0\}$ and each semicharacter is given by*

$$\sigma_{x,z}(s)=e^{i(x_1s_1+\cdots+x_ls_l+z_1s_{l+1}+\cdots+z_{n-l}s_n)}$$

where $x\in\mathbb{R}^l$, $z\in\mathbb{H}^{n-l}$ and $s=\sum_{j=1}^{n}s_jh_j\in S$.

Proof. **(i)** We may write $S\setminus\{0\}=\{s_1,s_2,\dots\}$ where $s_k<s_{k+1}$ for each k. We note that $d_k=\gcd(s_1,\dots,s_k)$ defines a decreasing sequence of natural numbers, hence there is n such that $d_n=d$. The "postage stamp problem" tells us that there is s_0 in $S\setminus\{0\}$ such that $S_0=\{s_0+kd:k\in\mathbb{Z}^{\geq 0}\}=\{s\in S:s\geq s_0\}$, i.e. if $d=m_1s_1+\cdots+m_ns_n$ where $m_1,\dots,m_n$ in $\mathbb{Z}$ are found by the Euclidean algorithm, then $s_0=(|m_1|+\cdots+|m_n|)s_1\dots s_n$ suffices.

Now let $\sigma\in\widehat{S}$. We note for any s,t in $S\setminus\{0\}$ we have $\sigma(t)^s=\sigma(st)=\sigma(s)^t$, hence if $\sigma(s)=0$ for any s in $S\setminus\{0\}$ then $\sigma|_{S\setminus\{0\}}=0$; let us assume otherwise. We define $\tau:\mathbb{N}\to\mathbb{C}$ by

$$\tau(k)=\frac{\sigma(s+kd)}{\sigma(s)}$$

for some s in S_0. This definition is independent of the choice of such s since $\sigma(s+kd)\sigma(t)=\sigma(s)\sigma(t+kd)$. Thus we find that for k,l in $\mathbb{N}$ that

$$\tau(k+l)=\frac{\sigma(s+kd+ld)}{\sigma(s)}\frac{\sigma(s+ld)}{\sigma(s+ld)}=\tau(k)\tau(l)$$

and hence $\tau(k)=\tau(1)^k$ for k in $\mathbb{N}$; let $z=\tau(1)$. Now for s in S_0 we write $s=kd$ for some k in $\mathbb{Z}^{\geq 0}$ and we have

$$\sigma(s)=\frac{\sigma(s+kd)}{\sigma(s)}=z^k=z^{s/d}.$$

If $s\in S\setminus(S_0\cup\{0\})$ let k be so $kd=s$. Then there is $m_0\in\mathbb{N}$ for which $m\geq m_0$ implies that $ms\in S_0$. For any such m, $\sigma(s)^m=\sigma(ms)=(z^{s/d})^m$, from which it follows that $\sigma(s)=z^{s/d}$.

(ii) Let $S_0=\{s\in S:\mathbb{R}^{\geq 1}s\in S\}$. It is clear that S_0 is a subsemigroup of S. Fix a norm $|\cdot|$ on $\mathbb{R}^n$ and for any x in $\mathbb{R}^n$ and $\delta>0$ let $B_\delta(x)$ denote the open δ-ball

about x. Since for t in S there is $\delta > 0$ so $B_\delta(t) \in S$, $\bigcup_{k=1}^{\infty} B_{k\delta}(kt) \subset S$, and we have

$$mt \in S_0 \text{ for } m \text{ in } \mathbb{N} \text{ with } m\delta \geq |t|. \tag{4.4}$$

Moreover S_0 is open. Indeed fix s_0 in S_0. By compactness, $\delta = \min\{\mathrm{dist}([1, 2 + |s_0|]s_0, \mathbb{R}^n \setminus S), 1\} > 0$, and it follows for s with $|s - s_0| < \delta/(1 + |s_0|)$, that $[1, 1 + |s|]s \in S$. Thus, by a straighforward tiling argument, $s \in \mathbb{R}^{\geq 1}s \subset S_0$.

Now let $D = \{h \in \mathbb{R}^n : s + \mathbb{R}^{\geq 0}h \in S_0 \text{ for all } s \text{ in } S_0\}$ It is clear that D is a semigroup of $\mathbb{R}^n$, $S_0 \subset D$ and $\mathbb{R}^{\geq 0}h \subset D$ for h in D. It follows that $D - D$ is a subgroup of $\mathbb{R}^n$ with non-empty interior, and hence all of $\mathbb{R}^n$, thus D contains a basis $\{h_1, \ldots, h_n\}$ for $\mathbb{R}^n$. We may re-order the basis and let l be so $\mathbb{R}h_j \subset D$ for $j = 1, \ldots, l$ and $\mathbb{R}h_j \not\subset D$ for $j = l + 1, \ldots, n$. We then find $D = \bigoplus_{j=1}^{l} \mathbb{R}h_j \oplus \bigoplus_{j=1}^{n-l} \mathbb{R}^{\geq 0}h_j$. We let H denote the interior of D. Now it is well-known that $\widehat{\mathbb{R}^{>0}} \cong \mathbb{H}$ via the identification $\sigma(t) = e^{itz}$. Since the formula $\widehat{S_1 \times S_2} \cong \widehat{S}_1 \times \widehat{S}_2$ holds, we find that $\widehat{H} = \mathbb{R}^l \times \mathbb{H}^{n-l}$ with duality as suggested above.

Now let $\sigma \in \widehat{S}$. Let $\tau : H \to \mathbb{C}$ be given by

$$\tau(h) = \frac{\sigma(s + h)}{\sigma(s)}$$

for any s in S_0 for which $\sigma(s) \neq 0$; by (4.4) such an s always exists. As in the proof of (i), above, $\tau \in \widehat{H}$ and is independant of choice of s. We note that $S \in H$. Indeed, if $t \in S$ pick m as in (4.4). If $s \in S_0$ and $\alpha \geq 1$ we have $\alpha(s+t) = \alpha s + \frac{\alpha}{m}mt$ and we note that $S_0 \in H$. It is immediate that for $t \in S$ that $\tau(t) = \sigma(s + t)/\sigma(s) = \sigma(t)$. □

Example 4.7. Let $G = \mathbb{T}$ and χ_n in $\widehat{\mathbb{T}}$ be given by $\chi_n(z) = z^n$ for z in $\mathbb{T}$. For any subset U of $\mathbb{Z}$, let $\lambda_U = \bigoplus_{n \in U} \chi_s$ where $\chi_s(z) = z^s$. Then

$$\mathrm{A}_{\lambda_U} = \left\{ z \mapsto \sum_{n \in U} \alpha_n z^n \text{ where } \sum_{n \in U} |\alpha_n| < \infty \right\} \cong \ell^1(U)$$

is a Banach space of Laurent polynomials. Let S be the subsemigroup generated by U. We have essentially two cases to consider.

(i) If S is a semigroup which contains both positive and negative elements, it is a group and hence of the form $\mathbb{Z}d$. In this case $\Phi_{\mathrm{A}(\lambda_S)} \cong \mathbb{T}$, via the character $\chi(z) = z^d$.

(ii) If $U = \{0, 1\}$ then $\tau_{\lambda_U} = \lambda_{\mathbb{Z}^{\geq 0}}$, and we find that $\mathcal{A}(\lambda_S) = \mathcal{A}_{\lambda_{\mathbb{Z}^{\geq 0}}} = \mathcal{A}(\mathbb{D})$ is the disc algebra, consisting of functions on $\mathbb{T}$ which are continuous and continuously extend to analytic functions on the interior of $\mathbb{D}$. If S is a subsemigroup of $\mathbb{Z}$ which is not a group, then we may suppose that $S \subset \mathbb{Z}^{\geq 0}$, otherwise take $-S$. Proposition 4.6 (i) shows that if U is any subset of $\mathbb{N}$, then $\mathcal{A}(\lambda_S)$ may be identified with the uniformly closed subalgebra of analytic functions on $\mathbb{D}$ generated by the monomials $z \mapsto z^n$ for $n \in U$.

(iii) Note that if we consider the topological semigroup $\mathbb{D}$ itself, it is an obvious consequence of the maximum modulus principle that the translation invariant algebra $\mathcal{A}(\mathbb{D})$ has Šilov boundary $\mathbb{T}$, which is not an ideal in $\mathbb{D}$. Thus Theorem 2.9 (iii) may not be true for any semitopological semigroup, without assuming the existence of a dense subgroup.

Example 4.8. Let $G = \mathbb{R}$ and χ_s in $\widehat{\mathbb{R}}$ be given by $\chi_s(t) = e^{ist}$.

(i) Let $\lambda_+ = \int_{\mathbb{R}^{>0}}^{\oplus} \chi_s \, dm(s)$ where m is Lebesgue measure. Then it is standard that $\mathrm{A}_{\lambda_+} = \mathrm{A}(\lambda_+) \cong \mathrm{L}^1(\mathbb{R}^{>0})$. Propositions 4.5 and 4.6 (ii) show that the Gelfand transform on $\mathrm{L}^1(\mathbb{R}^{>0})$ is given by the Laplace transform

$$\hat{f}(z) = \int_0^\infty f(t)e^{izt}dt$$

where $z \in \mathbb{H}$; it is clear that $\hat{f}$ is analytic on the interior of $\mathbb{H}$ and vanishes at ∞. Let $\mathcal{A}_0(\mathbb{H})$ denote the algebra of continuous functions on $\mathbb{R}$, which continuously extend to analytic functions on the interior of $\mathbb{H}$ which vanish at ∞ on all of $\mathbb{H}$. Note that $\Phi_{\mathcal{A}_0(\mathbb{H})} \cong \mathbb{H}$ and $\partial_{\mathcal{A}_0(\mathbb{H})} = \mathbb{R}$. Let us show that the uniform closure of $\{\hat{f}|_{\mathbb{R}} : f \in \mathrm{L}^1(\mathbb{R}^{>0})\}$ is $\mathcal{A}_0(\mathbb{H})$. First we consider the Cayley transform $\gamma : \mathbb{H} \to \mathbb{D} \setminus \{1\}$ given by $\gamma(z) = \frac{z-i}{z+i}$. The map $g \mapsto g\circ\gamma : (1-z)\mathcal{A}(\mathbb{D}) \to \mathcal{A}_0(\mathbb{H})$ is an isomorphism. If $g_n(z) = z^n - z^{n+1}$ then $g_n\circ\gamma(z) = \frac{2(z-i)^n}{i(z+i)^{n+1}}$. Let $f_n \in \mathrm{L}^1(\mathbb{R}^{>0})$ be given by $f_n(t) = t^n e^{-t}$. Then $\hat{f}_n(z) = \int_0^\infty t^n e^{i(z+i)t}dt = -\frac{i^n n!}{(z+i)^{n+1}}$. Hence it follows that $\mathrm{span}\{\hat{f}_k\}_{k=1}^n = \mathrm{span}\{g_k\circ\gamma\}_{k=1}^n$ for each n in $\mathbb{N}$, so $\{\hat{f} : f \in \mathrm{L}^1(\mathbb{R}^{>0})\}$ is dense in $\mathcal{A}_0(\mathbb{H})$.

(ii) If S is any open subsemigroup of $\mathbb{R}^{>0}$ we let $\lambda_S = \int_S^{\oplus} \chi_s \, dm(s)$. As above we get $\mathrm{A}_{\lambda_S} = \mathrm{A}(\lambda_S) \cong \mathrm{L}^1(S)$. The Gelfand transform on $\mathrm{L}^1(S)$, in this case, is a modified Laplace transform

$$\hat{f}(z) = \int_S f(t)e^{izt}dt.$$

By Proposition 4.6 (ii) and Theorem 2.9 (i), $\mathcal{A}(\lambda_S)$ is isomorphic to a $\mathbb{H}$-translation invariant subalgebra of $\mathcal{A}_0(\mathbb{H})$.

The simplest example is the semigroup $\mathbb{R}^{>a}$ where $a > 0$. By standard Laplace transform techniques we see that $\hat{f}(z) = e^{iaz}\int_0^\infty f(t-a)e^{izt}dt$ from which it follows that $\mathcal{A}(\lambda_{\mathbb{R}^{>a}}) \cong e^{iaz}\mathcal{A}_0(\mathbb{H})$.

If we let $\frac{1}{5} < a < \frac{1}{3}$, then the semigroup $S = \{s \in \mathbb{R} : 1-a < a < 1+a \text{ or } s > 2-2a\}$ can be shown, as above, to satisfy $\mathcal{A}(\lambda_S) = (e^{i(1-a)z} - e^{i(1+a)z} + e^{iaz})\mathcal{A}_0(\mathbb{H})$.

Notice that in both cases above $\mathcal{A}(\lambda_S)$ is a principal ideal in $\mathcal{A}(\lambda_{\mathbb{R}^{>0}})$.

(iii) In [27] many examples of the form $\pi = \bigoplus_{s\in S}\chi_s$, where S is a subsemigroup of $\mathbb{R}^{\geq 0}$, are given. These correspond to certain analytic semigroups which contain quotients of $\mathbb{R}^{\mathcal{AP}}$.

(iv) Consider the representation $\pi_+ \oplus \chi_1$. We have that $\tau_{\pi_+\oplus\chi_1} \cong_q \pi_+ \oplus \bigoplus_{n\in\mathbb{N}}\chi_n$ so

$$\mathrm{A}(\lambda_+ \oplus \chi_1) = \mathrm{A}_{\lambda_+} \oplus_{\ell^1} \mathrm{A}(\chi_1) \cong \mathrm{L}^1(\mathbb{R}^{>0}) \oplus_{\ell^1} \ell^1(\mathbb{N})$$

where $\ell^1(\mathbb{N})$ is the algebra of Dirac measures supported on $\mathbb{N}$. Taking uniform closure, we obtain a semidirect product algebra

$$\mathcal{A}(\lambda_+ \oplus \chi_1) = \mathcal{A}_0(\mathbb{H}) \oplus \mathcal{A}_0(\mathbb{D}) = \mathcal{A}_0(\mathbb{H} \sqcup \mathbb{D}_0)$$

where $\mathcal{A}_0(\mathbb{D}) = z\mathcal{A}(\mathbb{D})$. As suggested by Proposition 4.4, $\mathbb{H} \sqcup \mathbb{D}_0$ is a semigroup where $\mathbb{H}$ and $\mathbb{D}_0 = \mathbb{D}\setminus\{0\}$ are subsemigroups, and for z in $\mathbb{H}$ and w in $\mathbb{D}_0$ we define

$$zw = e^{iz}w = wz.$$

The topology is given by having $\mathbb{H}$ be open, and allowing neighbourhoods of elements w in $\mathbb{D}_0$ to be given by $U \sqcup V$ where V is a neighbourhood of w in $\mathbb{D}_0$, and U is a cocompact set in $\mathbb{H}$.

(v) Let $S = \mathbb{R}\times\mathbb{R}^{>0} \subset \mathbb{R}^2$. This is a subgroup for which we have

$$\mathcal{A}(\lambda_S) \cong \mathcal{C}_0(\mathbb{R})\check{\otimes}\mathcal{A}_0(\mathbb{H})$$

where $\check{\otimes}$ denotes the injective tensor product. Indeed, we have an isometric identification $\mathcal{C}_0(\mathbb{R})\check{\otimes}\mathcal{A}_0(\mathbb{H}) \cong \{u \in \mathcal{C}_0(\mathbb{R}^2) : u(x,\cdot) \in \mathcal{A}_0(\mathbb{H})\}$. Since $\mathrm{A}(\lambda_S) \cong \mathrm{L}^1(\mathbb{R}\times\mathbb{R}^{>0}) \cong \mathrm{L}^1(\mathbb{R})\hat{\otimes}\mathrm{L}^1(\mathbb{R}^{>0})$ (projective tensor product), it follows, in part from (i) above, that for each f in $\mathrm{L}^1(S)$, the "Laplace-Fourier transform" of f

$$\hat{f}(x,z) = \int_S f(s)e^{i(s_1x+s_2z)}\,ds$$

satisfies $f|_{\mathbb{R}^2} \in \mathcal{C}_0(\mathbb{R})\check{\otimes}\mathcal{A}_0(\mathbb{H})$, and the family of such functions is uniformly dense within.

Similarly, if $a_1,\dots,a_{n-l} \geq 0$ and $S = \mathbb{R}^l\times\mathbb{R}^{>a_1}\times\mathbb{R}^{>a_{n-l}} \subset \mathbb{R}^n$, then we can appeal to (ii) above and adapt the above methods to see that

$$\mathcal{A}(\lambda_S) \cong \mathcal{C}_0(\mathbb{R}^k)\check{\otimes}e^{ia_1z}\mathcal{A}_0(\mathbb{H})\check{\otimes}\dots\check{\otimes}e^{ia_{n-l}z}\mathcal{A}_0(\mathbb{H}).$$

Example 4.9. We note some recent results of [16]. For $G = \mathbb{Z}$, there is a representation π for which $G^\pi \cong \mathrm{L}^\infty[0,1]_{\|\cdot\|\leq 1}$.

Furthermore, if $G = (\mathbb{Z}/p\mathbb{Z})^{\oplus\infty}$, then there is a representation π for which $G^\pi \cong \{\varphi \in \mathrm{L}^\infty[0,1] : \operatorname{ess\,ran}\varphi \subset \Gamma_p\}$ where $\Gamma_p = \operatorname{conv}\{e^{2\pi ik/p} : k = 0,\dots,p-1\}$.

More significantly, for each example above there are $\mathfrak{c}$ idempotents in G^π, and it is shown the closure of this family is isomorphic to $\{\varphi \in \mathrm{L}^\infty[0,1] : \operatorname{ess\,ran}\varphi \subset [0,1]\}$.

4.3. **Compact matrix groups.** Let G be a compact group, and σ be a continuous finite dimensional unitary representation, so $G^\sigma = \sigma(G)$ may be regarded as a closed subgroup of the unitary group $\mathcal{U}(\mathcal{H}_\sigma) \subset \mathcal{B}(\mathcal{H}_\sigma)$.

We now use some ideas form Lie theory. We let

$$\mathfrak{g}^\sigma = \{X \in \mathcal{B}(\mathcal{H}_\sigma) : \exp(tX) \in G^\sigma \text{ for all } t \text{ in } \mathbb{R}\}.$$

It is well-known that $\mathfrak{g}^\sigma$ is a real Lie algebra with $[X,Y] = XY - YX$. Moreover $\mathfrak{g}^\sigma$ has the same reducing subspaces as σ, so $\mathfrak{g}^\sigma \subset \mathrm{VN}_\sigma$. We let $\mathfrak{g}^\sigma_{\mathbb{C}} = \mathfrak{g}^\sigma + i\mathfrak{g}^\sigma$ denote its complexification and

$$G^\sigma_{\mathbb{C}} = G^\sigma\langle\exp\mathfrak{g}^\sigma_{\mathbb{C}}\rangle$$

which is a complex Lie subgroup of invertible elements in VN_σ. We then let $\mathbb{D}^\sigma = \overline{G^\sigma_{\mathbb{C}}} \cap \mathcal{B}(\mathcal{H}_\sigma)_{\|\cdot\|\leq 1}$

Theorem 4.10. *We have $\Phi_{\mathrm{A}(\sigma)} \cong \mathbb{D}^\sigma \setminus \{0\}$.*

Proof. For this proof, we shall make use of the Zariski topolgy on the finite dimensional affine space $\mathcal{B}(\mathcal{H}_\sigma)$. We let $\mathrm{pol}(\mathcal{B}(\mathcal{H}_\sigma))$ denote the algebra generated by the matrix coefficient functionals and the constant functional 1. Then for $S \subset \mathcal{B}(\mathcal{H}_\sigma)$, we let $\mathrm{i}(S) = \{p \in \mathrm{pol}(S) : p|_S = 0\}$ and let the Zariski closure of S be given by

$$Z(S) = \bigcap_{p\in\mathrm{i}(S)} p^{-1}(\{0\}).$$

We observe that $Z(\{0\}) = \{0\}$ and $Z(S\cup\{0\}) = Z(S)\cup\{0\}$ since $S \mapsto Z(S)$ is a closure operation. We also note that $Z(S)$ is the largest set Z for which $\mathrm{i}(S) = \mathrm{i}(Z)$ and hence the spectrum of the algebra $\mathrm{pol}(\mathcal{B}(\mathcal{H}_\sigma))|_S \cong \mathrm{pol}(\mathcal{B}(\mathcal{H}_\sigma))/\mathrm{i}(S)$ is naturally identified with $Z(S)\setminus\{0\}$. Thus, recognising $\mathrm{alg}(F_\sigma)$ as the algebra $\mathrm{pol}(\mathcal{B}(\mathcal{H}_\sigma))|_{G^\sigma\cup\{0\}}$, we obtain spectrum

$$\Phi_{\mathrm{alg}(F_\sigma)} \cong Z(G^\sigma\cup\{0\})\setminus\{0\} = Z(G^\sigma)\setminus\{0\}. \tag{4.5}$$

We next wish to establish that $G^\sigma_{\mathbb{C}} \subset Z(G^\sigma)$. First, if $p \in \mathrm{i}(G^\sigma)$, we have that $p\circ\exp : \mathfrak{g}^\sigma_{\mathbb{C}} \to \mathbb{C}$ is a holomorphic function, which vanishes on $\mathfrak{g}^\sigma$, a real subspace whose complex span is $\mathfrak{g}^\sigma$. Hence $\exp(\mathfrak{g}^\sigma_{\mathbb{C}}) \subset Z(G^\sigma)$. By virtue of the fact that G^σ is a (semi)group, we have for p in $\mathrm{i}(G^\sigma)$ that $s\cdot p \in \mathrm{i}(G^\sigma)$ for s in G^σ. Hence for v in $Z(G^\sigma)$, $p(vs) = 0$ for p and s as above, and we have that $p\cdot v \in \mathrm{i}(G^\sigma)$. Thus for w in $Z(G^\sigma)$ we find $p(vw) = p\cdot v(w) = 0$, so $vw \in Z(G^\sigma)$. Thus $Z(G^\sigma)$ is a semigroup, and it follows that $G^\sigma_{\mathbb{C}} = G^\sigma\langle\exp(\mathfrak{g}^\sigma_{\mathbb{C}})\rangle \subset Z(G^\sigma)$.

We now wish to establish that $G^\sigma_{\mathbb{C}}$ is Zariski closed in $\mathcal{B}(\mathcal{H}_\sigma)_{inv}$. To see this we consider $G^{\sigma\oplus\bar\sigma}_{\mathbb{C}}$. We observe that

$$G^{\sigma\oplus\bar\sigma} \cong \left\{ \begin{bmatrix} u & 0 \\ 0 & \bar u \end{bmatrix} : u \in G^\sigma \right\} \cong G^\sigma$$

Hence, following calculations such as in [44, Cor. 2.2], we see that

$$G^{\sigma\oplus\bar\sigma}_{\mathbb{C}} \cong \left\{ \begin{bmatrix} v & 0 \\ 0 & v^{-T} \end{bmatrix} : v \in G^\sigma_{\mathbb{C}} \right\} \cong G^\sigma_{\mathbb{C}}$$

where v^{-T} is the inverse transpose of v. We also note that we can identify $\mathrm{alg}(F_{\sigma\oplus\bar\sigma})$ with the algebra of trigonometric functions on G^σ, thanks to [32, (27.39)]. Then $G^\sigma_{\mathbb{C}}$ is naturally identified with the spectrum of $\mathrm{alg}(F_{\sigma\oplus\bar\sigma})$, [8, (2.3) & Thm. 2]. Combining this with (4.5) we obtain that $G^{\sigma\oplus\bar\sigma}_{\mathbb{C}} = V(G^{\sigma\oplus\bar\sigma})$. The same polynomial equations, on matrices in the upper left corner, establish that $G^\sigma_{\mathbb{C}}$ is Zariski closed in $\mathcal{B}(\mathcal{H}_\sigma)_{inv}$.

It follows from the fact above that $G^\sigma_{\mathbb{C}} = Z(G^\sigma_{\mathbb{C}}) \cap \mathcal{B}(\mathcal{H}_\sigma)_{inv} = Z(G^\sigma) \cap \mathcal{B}(\mathcal{H}_\sigma)_{inv}$. However, since $\mathcal{B}(\mathcal{H}_\sigma)_{inv}$ is Zariski open, we can use [50, I.10 Thm. 1], to establish that

$$\overline{G^\sigma_{\mathbb{C}}} = \overline{Z(G^\sigma) \cap \mathcal{B}(\mathcal{H}_\sigma)_{inv}} = Z(G^\sigma).$$

Finally, we note that each element of $\Phi_{\mathrm{A}(\sigma)}$ is contractive and determined by its restriction to the dense subalgebra $\mathrm{alg}(F_\sigma)$; while each contractive element of $\overline{G^\sigma_{\mathbb{C}}} \cong \Phi_{\mathrm{alg}(F_\sigma)}$ extends to a character on $\mathrm{A}(\sigma)$. □

We note that by [47, Cor. 1], each v in $G^\sigma_{\mathbb{C}}$ admits a polar decomposition $u|v|$ where $u \in G^\sigma$ and hence $v \in G^\sigma_{\mathbb{C}}$. This corresponds to the polar decomposition observed in Theorem 3.6.

Example 4.11. (i) Let $G = \mathrm{U}(d)$, the group of $d\times d$-unitary matrices, and let $\sigma : \mathrm{U}(d) \to \mathcal{U}(\mathbb{C}^d)$ denote the standard representation, so $G^\sigma = \mathcal{U}(\mathbb{C}^d)$. It is well-known that the Lie algebra is $\mathfrak{u}(d) = \{X \in \mathcal{B}(\mathbb{C}^d) : X^* = -X\}$, whence $\mathfrak{u}(d)_{\mathbb{C}} = \mathcal{B}(\mathbb{C}^d)$, and hence $\mathcal{U}(\mathbb{C}^d)_{\mathbb{C}} = \mathcal{B}(\mathbb{C}^d)_{inv}$. This space is dense in $\mathcal{B}(\mathbb{C}^d)$, so by Theorem 4.10 $\Phi_{\mathrm{A}(\sigma)} = \mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1} \setminus \{0\}$.

It is standard that the convex hull of $\mathcal{U}(\mathbb{C}^d)$ is $\mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1}$, i.e. we have for v in $\mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1}$ polar decomposition $v = u|v| = \frac{1}{2}u[(|v| + i\sqrt{1-|v|^2}) + (|v| - i\sqrt{1-|v|^2})]$. Hence, each element of $\mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1}$ may be veiwed as a convex combination of elements $\sigma^\sigma_e(\varepsilon_{\mathcal{A}(\sigma)}(u))$ — see notation of Corollary 3.7 — for u in $\mathrm{U}(d)$. It follows that $\Phi_{\mathcal{A}}(\sigma) \cong \mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1}$, thus verifying Conjecture 3.9 in this case. We obseve that $\mathcal{A}(\sigma) \cong \mathcal{A}_0(\mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1})$, the "punctured ball algebra", i.e. the algebra of all continuous functions on $\mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1}$ which are holomorphic on the interior and vanish at 0. Similarly we find that $\mathcal{A}(\sigma\oplus 1) \cong \mathcal{A}(\mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1})$, the ball algebra. It is well-known that the Šilov boundary of $\mathcal{A}(\mathcal{B}(\mathbb{C}^d)_{\|\cdot\|\leq 1})$ is $\mathcal{U}(\mathbb{C}^d)$, see, for example [19, §12.4].

(ii) Let H denote either of the classical compact matrix groups $\mathrm{O}(d)$ or $\mathrm{Sp}(d)$. Let $G = \mathbb{T}\cdot H = \{zu : z \in \mathbb{T} \text{ and } u \in H\}$, and $\sigma : G \to \mathcal{B}(\mathbb{C}^{d'})$ be the standard representation where $d' = d$ in the case that $H = \mathrm{O}(d)$, and $d' = 2d$ in the case that $H = \mathrm{Sp}(d)$. It is straightforward to compute that $G^\sigma_{\mathbb{C}} = \mathbb{C}^{\neq 0}\cdot H_{\mathbb{C}} = \{\alpha v : \alpha \in \mathbb{C}^{\neq 0} \text{ and } v \in H_{\mathbb{C}}\}$. We note that $\mathrm{O}(d)_{\mathbb{C}} = \mathrm{O}(d,\mathbb{C})$ and $\mathrm{Sp}(d)_{\mathbb{C}} = \mathrm{Sp}(d,\mathbb{C})$ are each closed algebraic groups.

We claim that $\overline{G^\sigma_{\mathbb{C}}} = \mathbb{C}\cdot H_{\mathbb{C}}$. Indeed, if $\lim_{n\to\infty} \alpha_n v_n = b$ then we factor $\alpha_n v_n = (\alpha_n \|v_n\|)(\frac{1}{\|v_n\|}v_n)$. By dropping to a subsequence, we may suppose that $\lim_{n\to\infty} \alpha_n \|v_n\| = \alpha$ and $\lim_{n\to\infty} \frac{1}{\|v_n\|}v_n = v \in H_{\mathbb{C}}$. Hence $b = \alpha v \in \mathbb{C}\cdot H_{\mathbb{C}}$. Thus $\mathbb{D}^\sigma = \mathbb{C}\cdot H_{\mathbb{C}} \cap \mathcal{B}(\mathbb{C}^{d'})_{\|\cdot\|\leq 1}$.

Thus $\Phi_{\mathrm{A}(\sigma)} \cong \mathbb{D}^\sigma \setminus \{0\} = G^\sigma_{\mathbb{C}} \cap \mathcal{B}(\mathbb{C}^{d'})_{\|\cdot\|\leq 1}$. We have not devised a means to show that $\Phi_{\mathcal{A}(\sigma)} = \Phi_{\mathrm{A}(\sigma)}$ in either of these cases.

(iii) If $\sigma : G \to \mathcal{U}(\mathcal{H}_\sigma)$ is an injective homomorphism, then $\rho_\sigma \cong_q \lambda$, the left regular representation, by [32, (27.39)] and the Peter-Weyl Theorem. Thus $\mathrm{E}(\sigma) = \mathrm{A}(G)$ and $\Phi_{\mathrm{E}(\sigma)} = G$.

In particular if $G = \mathrm{SU}(2)$ and $\sigma : \mathrm{SU}(2) \to \mathrm{U}(2)$ is the standard representation, then $\bar\sigma \cong \sigma$ and it follows that $\mathrm{A}(\sigma) = \mathrm{E}(\sigma) = \mathrm{A}(\mathrm{SU}(2))$.

4.4. **A non-compact, non-abelian example.**

Example 4.12. Let G be the $ax+b$-group, given by

$$\{(a,b) : a \in \mathbb{R}^{>0}, b \in \mathbb{R}\}$$

with multiplication $(a,b)(a',b') = (aa', ab'+b)$. Let us consider, adapted from the notation of [21, p. 189] (we omit normalisation by 2π), the representation $\pi_+ : G \to \mathcal{B}(\mathrm{L}^2(\mathbb{R}^{>0}, m))$ given by

$$\pi_+(a,b)f(s) = a^{1/2}e^{ibs}f(as) \tag{4.6}$$

for f in $\mathrm{L}^2(\mathbb{R}^{>0}, m)$ and m-a.e. s in $\mathbb{R}^{>0}$, where m is the usual Lebesgue measure. We wish to compute $\Phi_{\mathcal{A}(\pi_+)}$.

According to [40, Théo. 5], $\mathrm{A}_{\pi_+} = \mathrm{A}(\pi_+)$. Implicit in the proof of that fact, see [40, p. 159] is a formula for $\pi_+ \otimes \pi_+$. We rederive this formula in a form more tractable to to obtaining (4.8). For convenience we write $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R}^{>0}, m)$, which we identify as the subspace $\mathrm{L}^2(\mathbb{R})1_{\mathbb{R}^{>0}}$ of $\mathrm{L}^2(\mathbb{R})$. We consider the direct integral Hilbert space $\mathcal{H} = \int^\oplus_{\mathbb{R}^{>0}} \mathrm{L}^2_t\, dt$ where each L^2_t is a copy of L^2. We identify $\mathrm{L}^2 \otimes^2 \mathrm{L}^2$ with $\mathrm{L}^2(\mathbb{R}^2)1_{(\mathbb{R}^{>0})^2}$ in the usual manner, and define $U : \mathrm{L}^2 \otimes^2 \mathrm{L}^2 \to \mathcal{H}$ by

$$U\xi = \int^\oplus_{\mathbb{R}^{>0}} (\lambda(t)\otimes I)\xi(\cdot,t)\, dt$$

where $\lambda(t)f(s) = f(t-s)$ for $f \in \mathrm{L}^2(\mathbb{R})1_{\mathbb{R}^{>0}} \cong \mathrm{L}^2(\mathbb{R}^{>0})$. It is straightforward to verify that U is a unitary with $U^*\left(\int^\oplus_{\mathbb{R}^{>0}} u_{t'}\, dt'\right)(s,t) = u_t(s+t)$. We define for (a,b) in G, $\Pi_+(a,b) : \mathcal{H} \to \mathcal{H}$ by

$$\Pi_+(a,b)\int^\oplus_{\mathbb{R}^{>0}} u_t\, dt = a^{1/2}\int^\oplus_{\mathbb{R}^{>0}} \pi_+(a,b)u_{at}\, dt.$$

We have for f, g, h, k in L^2 that

$$
\begin{aligned}
\langle \pi_+ \otimes \pi_+(a,b) f\otimes g | h\otimes k\rangle &= \langle \pi_+(a,b)f|h\rangle \, \langle \pi_+(a,b)g|k\rangle \\
&= a\int_{\mathbb{R}^{>0}}\int_{\mathbb{R}^{>0}} e^{ib(s+t)} f(as)g(at)\overline{h(s)k(t)}\,dt\,ds \\
&= a^{1/2}\int_{\mathbb{R}^{>0}}\int_{\mathbb{R}^{>0}} a^{1/2}e^{ibs} f(a(s-t))g(at)\overline{h(s-t)k(t)}\,ds\,dt \\
&= a^{1/2}\int_{\mathbb{R}^{>0}} \langle \pi_+(a,b)\lambda(at)f\, g(at)|\lambda(t)h\, k(t)\rangle\; dt \\
&= \langle \Pi_+(a,b)Uf\otimes g|Uh\otimes k\rangle\,.
\end{aligned}
\tag{4.7}
$$

In particular Π_+ is a representation, unitarily equivalent to $\pi_+\otimes\pi_+$. Thus Theorem 3.6 (3.3) tells us that for x in VN_π we have that $x\in\Phi_{\mathrm{A}(\pi_+)}\cup\{0\}$ if and only if

$$
U^*\Pi_+^{\pi_+}(x)U = x\otimes x. \tag{4.8}
$$

We remark that since π_+ is irreducible, $\mathrm{VN}_{\pi_+}=\mathcal{B}(\mathrm{L}^2)$. We let

$$
\tilde{G} = \{(a,z) =: a\in\mathbb{R}^{>0}, z\in\mathbb{H}\}
$$

which is a semigroup via $(a,z)(a',z') = (aa', az'+z)$. For (a,z) in $\tilde{G}$, define $\tilde{\pi}_+(a,z)$ exactly as in (4.6). A straightforward repeat of (4.7) shows that each $\tilde{\pi}_+(a,z)$ satsfies (4.8). We conjecture that $\Phi_{\mathrm{A}(\pi_+)}\cong\tilde{G}$. We observe that the conjugation $(a,z)^* = (a^{-1}, -a^{-1}\bar{z})$ satisfies $\tilde{\pi}_+((a,z)^*) = \tilde{\pi}_+(a,z)^*$. We have that $(a,z)^*(a,z) = (1, 2a^{-1}i\mathrm{Im}z)$. It is clear that $\tilde{\pi}_+(1,2a^{-1}i\mathrm{Im}z)$ is the operator of multiplication by $x\mapsto e^{-2a^{-1}\mathrm{Im}z\,x}$ on L^2, whose positive square root is the operator of multiplication by $x\mapsto e^{-a^{-1}\mathrm{Im}z\,x}$, i.e. $|\tilde{\pi}_+(a,z)| = \tilde{\pi}_+(1,a^{-1}i\mathrm{Im}z)$. Thus we obtain a formula for the polar decomposition from Theorem 3.6: $(a,z) = (a,\mathrm{Re}z)(1,a^{-1}i\mathrm{Im}z)$.

We now claim that

$$
\mathcal{A}(\pi_+)\cong\mathcal{C}_0(\mathbb{R}^{>0})\check{\otimes}\mathcal{A}_0(\mathbb{H}) \tag{4.9}
$$

where $\check{\otimes}$ denotes the injective tensor product and $\mathcal{A}_0(\mathbb{H})$ is defined in Example 4.8 (i). Indeed, $\mathcal{A}(\pi_+)\subset\mathcal{C}_0(G)\cong\mathcal{C}_0(\mathbb{R}^{>0}\times\mathbb{R})$ is a point-separating subalgebra which has the span of functions $(a,b)\mapsto a^{1/2}\int_{\mathbb{R}^{>0}} e^{ibs}f(as)\overline{g(s)}\,ds$ where $f,g\in\mathrm{L}^2$. For fixed b, such a function is easily seen to be a generic element of $\mathrm{A}(\mathbb{R}^{>0})$, the space of which is dense in $\mathcal{C}_0(\mathbb{R}^{>0})$. For a fixed a, such a function may be seen to be the Laplace transform $b\mapsto\hat{h}(b)$, of a generic element h in $\mathrm{L}^1(\mathbb{R}^{>0},m)$, a set of elements which is dense in $\mathcal{A}_0(\mathbb{H})$, as demonstrated in Example 4.8 (i).

We note that it is immediate that $\Phi_{\mathcal{A}(\pi_+)}\cong\tilde{G}$. Conjecture 3.9 if true, would now tell us that $\Phi_{\mathrm{A}(\pi_+)}\cong\tilde{G}$.

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